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Th Anderson

AN

ELEMENTARY TREATISE

ON

ALGEBRA,

THEORETICAL AND PRACTICAL;

WITH

ATTEMPTS TO SIMPLIFY SOME OF THE MORE DIFFICULT PARTS OF THE SCIENCE,

PARTICULARLY

THE DEMONSTRATION OF THE BINOMIAL THEOREM

IN ITS MOST GENERAL FORM;

THE SUMMATION OF INFINITE SERIES;

THE SOLUTION OF EQUATIONS OF THE HIGHER ORDER, &c.

FOR THE USE OF STUDENTS.

BY J. R. YOUNG,

PROFESSOR OF MATHEMATICS IN THE ROYAL COLLEGE, BELFAST,

And Author of "Elements of Geometry;" "Elements of Mechanics;" "Elements of Plane and Spherical Trigonometry;" "Mathematical Tables;" "Computation of Logarithms;" "Elements of Analytical Geometry;" "Elements of the Differential Calculus;" and "Elements of the Integral Calculus."

A NEW AMERICAN,

FROM THE LAST LONDON EDITION,

Revised and corrected by a Mathematician of Philadelphia.

PHILADELPHIA:

CAREY, LEA & BLANCHARD.

1838.

STEREOTYPED BY J. FAGAN......PHILADELPHIA.

PRINTED BY T. K. AND P. G. COLLINS.

TO

OLINTHUS GREGORY, ESQ., LL.D. F.R.A.S.

Corresponding Associate of the Academy of Dijon, Honorary Member of the Literary and Philosophical Society of New York; of the New York Historical Society; of the Literary and Philosophical, and the Antiquarian Societies of Newcastle-upon-Tyne; of the Cambridge Philosophical Society; of the Institution of Civil Engineers, &c. &c.; and Professor of Mathematics in the Royal Military Academy.

SIR,

IN permitting me to dedicate to you the following pages, you have conferred upon me an honour for which I feel truly grateful.

Your profound attainments as a Mathematician and Philosopher are so universally acknowledged, and so highly appreciated, that any production, however humble, which is introduced to the world under the sanction of your good opinion, will be considered as entitled to some degree of attention.

With these claims to notice, the present performance possesses advantages which I did not originally presume to anticipate, and, in laying it before you, allow me to assure you that

I am, Sir,

With the profoundest respect,

Your most obliged and most obedient servant,

THE AUTHOR.

PREFACE TO THE ENGLISH EDITION.

THE first edition of the present work was printed in octavo, and published at a price too high to warrant any very sanguine expectations as to the extent of its circulation. It gradually, however, found its way into the principal educational establishments of this kingdom—was adopted in the colleges of the United States*—in the African College at the Cape of Good Hope—and in New South Wales. This extensive patronage, unexpected alike by the publisher and myself, I attribute—not to any novelties contained in the book, but entirely to the efforts I made to simplify as much as possible the more difficult parts of the subject, and thus to present to the young mathematical student a clear and perspicuous view of the fundamental principles of analytical calculation. Many complaints have, however, reached me from mathematical teachers, to the effect that the practical examples were not found sufficient in number fully to illustrate the theory. My own experience has proved to me the justness of this complaint, and has, moreover, led me to detect several other blemishes in the work. In this new edition, it is hoped that these faults will be found in a great measure to have disappeared. The practical part has been considerably augmented throughout, the theory corrected and improved, and several new and interesting topics added. One subject, touched upon in the former edition, it has been thought advisable to exclude from this,—the chapter on the Theory of the Higher Equations; but the exclusion which has been thus made, as well as the additional matter which has been introduced, seemed equally necessary, to render the book better adapted to the wants of beginners, and more suitable for junior mathematical classes, in places of public education. Besides, in the progress of any science towards perfection, some departments of it are always found to receive more from the contributions of time than others; these departments gradually increase in magnitude and importance, till at length, detaching themselves from the main body, they become objects of individual importance and of distinct attention. Such has been the case with the Theory of Equations. That it is strictly a branch of pure Algebra there is no doubt; but it has exercised the talents and received the contributions of so many great men, and

* An American reprint, ably edited by Mr. Ward, of Columbia College, was published, in 1832, at Philadelphia.

has, consequently, at length acquired such extent and importance, as to have assumed the form of a distinct department of analysis. The discussion of this subject is therefore reserved for a separate volume, now at press, which will form a supplement to the present treatise.

The following brief enumeration of the principal topics discussed in this work is extracted, with slight modifications, from the preface to the former edition.

Chapter I. contains the Preliminary Rules of the science, in which the fundamental principles of operation are explained and illustrated.

Chap. II. is on Simple Equations, and commences with some propositions preparatory to entering upon the solution of an equation, which operation they are intended to render more easy and inviting. Then follow the several methods of solving simple equations involving one, two, and three or more unknown quantities; each of these methods being illustrated separately, not only by algebraical examples, but also by practical questions; a mode rather different from that usually adopted, but which appears to be preferable, as it affords the student an early opportunity of applying the principles that he has acquired to useful and interesting inquiries, an exercise which is generally found to be peculiarly pleasing and encouraging.

Chap. III. treats of Ratio, Proportion, and Progression, both arithmetical and geometrical; and, although the general formulas are fewer in number than those given in most books on this subject, yet it is shown that they are amply sufficient for every variety of case, and that therefore it would be superfluous to extend their number.

Chap. IV. is on Quadratics, and on Imaginary Quantities. This chapter is of a more difficult nature than either of the preceding, and proportionate pains have been taken to render the modes of operation clear and intelligible; the solutions to some of the more difficult examples, which are given at length, will be of service to the student in cases of a similar nature, and will manifest to him how much a little judgment and ingenuity on his part will add to the elegance of his operation. The article on Imaginary Quantities, with which this chapter concludes, will be found to contain some observations tending to remove the obscurity in which the subject is usually enveloped.

Chap. V. contains the general investigation of the Binomial Theorem. The demonstration of this celebrated Theorem in a manner adapted to elementary instruction, has always been considered as an object greatly to be desired, and many attempts have accordingly been made by different mathematicians for this purpose; all, however, that have yet appeared have been objected to, either on account of unwarrantable assumptions at the outset, which have consequently weakened the evidence, and rendered the demonstration incomplete, or because of a too tiresome and obscure method

of reasoning, which has been incomprehensible to a learner. The demonstration given in this chapter is, I believe, different from any that has been previously offered, and appears to be more simple and satisfactory than any which I have had an opportunity of seeing. In the practical application of this theorem to the expansion of a binomial, it is always best to separate the case in which the exponent is *integral*, from that in which it is *fractional*, because, in the former instance, the process by the general formula is unnecessarily long and troublesome; a different method of proceeding is therefore usually pointed out; but it is rather singular that it has been applied only when the exponent is a *positive* integer: as, however, it is equally applicable when the exponent is a *negative* integer, it is here extended to that case.

Chap. VI. explains the nature and construction of Logarithms, and shows their importance in their application to several useful inquiries relating to interest, annuities, &c.

Chap. VII. is devoted to Series; and a new method for the summation of infinite series is given, which it is thought will be found to be more direct and easy than those generally used in elementary works. Several interesting subjects connected with series will be found in this chapter.

Chap. VIII. is on Indeterminate Equations of the first degree. In this chapter also some improvements will be found. The rule given at page 234 for solving an indeterminate equation involving two unknown quantities, is more direct and concise than the usual method, and equally simple.

Chap. IX. contains the principles of the Diophantine Analysis, or of indeterminate equations above the first degree, and concludes with a collection of diophantine questions; several of which are solved, in order to exhibit to the student the artifices which are sometimes to be employed in this part of the subject. This chapter has little claim to novelty, except as far as relates to the introduction of some new questions, and to the new solutions given to others.

From the above outline an idea may be formed of the nature and pretensions of the work here submitted to the judgment of an impartial public; and if, upon examination, it shall be found that I have at all succeeded in my endeavours to lessen the labours of the student, it will afford me the highest satisfaction.

J. R. YOUNG.

ROYAL COLLEGE, BELFAST;
August 19th, 1834.

ADVERTISEMENT.

THE American publishers of Professor Young's Elementary Treatise on Algebra, with the desire of rendering the work as useful and correct as possible, have been induced to submit it to the perusal of a competent mathematician. At his suggestion, they have retained the Theory of Equations, which formed a part of the first London edition, and which they have judged to be a necessary supplement to the present volume. From the care taken by the reviser, and the number of corrections made, they are confident that exceedingly few, if any errors, could have escaped notice, and they accordingly offer the work thus improved to the attention of the public.

PHILADELPHIA, *April*, 1838.

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AN

ELEMENTARY TREATISE

ON

ALGEBRA.

CHAPTER I.

ON THE PRELIMINARY RULES OF THE SCIENCE.

Definitions.

(*Article* 1.) ALGEBRA is that branch of Mathematics, which teaches the method of performing calculations by means of letters and signs—the letters being employed to represent *quantities*, and the signs to represent the *operations* performed on them.

(2.) The sign + (*plus*) denotes addition.

— (*minus*) denotes subtraction.

Thus, $a + b$ signifies that the quantity represented by b is to be *added* to that represented by a; and $a - b$ signifies that the quantity represented by b is to be *subtracted* from that represented by a. If for instance, a represent 10, and b 2, then $a + b$ is 12, and $a - b$ is 8.

(3.) $\times$ (*Into*) denotes multiplication.

$\div$ (*Divided by*) signifies that the former of the two quantities, between which it is placed, is to be divided by the latter.

Thus, $a \times b$ signifies that the quantity represented by a is to be *multiplied* by that represented by b; and $a \div b$ signifies that a is to be *divided* by b. Multiplication is also denoted, sometimes by a dot placed between the quantities to be multiplied, as $a \, . \, b$, or without

any sign, as simply $a\,b$, each of which is the same as $a \times b$. Division also is often denoted by placing the dividend over the divisor, and drawing a line between: thus, $\frac{a}{b}$ is the same as $a \div b$.

(4.) $=$ (*Equal to*) denotes an equality of the quantities between which it is placed.

Thus, $a + b = 12$ signifies that a plus b is equal to 12; and $a - b = 8$, signifies that a minus b is equal to 8; also, $a + c - d = b + e$, denotes an equality between $a + c - d$ and $b + e$.

(5.) The *power* of any quantity is that quantity multiplied any number of times by itself. Thus, $a \times a$ is the second power of a, and is expressed in this manner, a^2; also, $a \times a \times a$ is the third power of a, and is expressed thus, a^3; likewise a^4 is the fourth power of a; a^5 the fifth power of a, &c.

(6.) The *root* of any quantity is a quantity which, if multiplied by itself a certain number of times, produces the original quantity; and it is called the *second* root, *third* root, &c. according to the number of multiplications. Thus the *second* or *square* root of a is a quantity whose *square* or *second power* produces a; the *third* or *cube root* of a is a quantity whose *cube* or *third power* produces a; the *fourth root* of a is a quantity whose *fourth power* produces a, &c. *Roots* are represented thus: $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c. or $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, &c. either of which forms respectively represents the *square root*, *cube root*, *fourth root*, &c. In the case of the *square root*, however, the 2 above the *radical sign* $\surd$ is usually omitted. Suppose $a = 16$, then $\sqrt{a}$ or $a^{\frac{1}{2}} = 4$, because 4×4, or $4^2 = 16$, also $\sqrt[4]{a}$, or $a^{\frac{1}{4}} = 2$, because $2 \times 2 \times 2 \times 2$, or $2^4 = 16$.

(7.) If unity be divided by any *power* or *root* of a quantity, as $\frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^{\frac{1}{2}}}$, &c., it is also expressed thus: a^{-2}, a^{-3}, $a^{-\frac{1}{2}}$, &c. The small figures used to denote powers or roots are called *indices* or *exponents*.

(8.) A *simple* quantity is that which consists of but one term, as a, ab, $4bc$, &c.

(9.) A *compound* quantity consists of two or more *simple* quantities, as $a + b$, $3ab - 2ad + e$, &c. If a compound quantity consist of two terms, it is called a *binomial*; if of three terms, a *tri-*

nomial; if of four terms, a *quadrinomial;* and if of more than four, a *polynomial* or *multinomial.*

(10.) The *coefficient* of a quantity is the number prefixed to it to denote how many times it is to be taken. Thus, $5x$ signifies five times the quantity x, and the number 5 is the coefficient of x. Also, in the expressions, $3xy$, $4abc$, $7yz$, &c., the coefficients are severally 3, 4, and 7; when no number is prefixed, as is the case when the quantity is to be taken but *once*, we say that the coefficient is unity; the quantities xy, abc, yz, &c. are in fact the same as $1xy$, $1abc$, $1yz$, the 1 being understood although it does not appear.

It is sometimes found convenient, when the coefficients are large numbers, to represent them, as well as the quantities which they multiply, by letters; choosing always, agreeably to art. (1), the leading letters of the alphabet for this purpose; and hence arises the verbal distinction between *numeral* coefficients and *literal* coefficients. Thus, if we agree to represent the number 46852 by a, then $46852\,x$ may be more briefly written ax, where x has the literal coefficient a.

(11.) *Like* terms are those of which the *literal* parts, disregarding the coefficients, are the same; that is, however their coefficients may differ, the quantities to which they are prefixed are all alike. Thus the following terms with numeral coefficients are *like terms*, viz. $4ax$, $7ax$, $3ax'ax$, &c.; and so are these with literal coefficients: az, bz, cz, &c.

(12.) *Unlike* terms are those which consist of *different letters*, as the terms $4ab$, $7cd$, ef, &c.

(13.) A *vinculum* or bar ———, or a parenthesis ( ), is used to connect several quantities together: Thus, $\overline{a+x} \times b$, or $(a+x) \times b$, signifies that the compound quantity $a+x$ is to be multiplied by b, also, $\overline{2ac+3b} \times \overline{4ax-2by}$ signifies that $2ac+3b$ is to be multiplied by $4ax-2by$. The bar is also sometimes placed vertically, thus:

$$\left.\begin{array}{r} a \\ -\,b \\ +\,c \end{array}\right| x \quad \text{is the same as } (a-b+c)\,x, \text{ or } \overline{a-b+c}\,.\,x.$$

(14.) To avoid the too frequent repetition of the word *therefore*, or *consequently*, the sign $\therefore$ is sometimes used.

NOTE. Quantities with the sign + are called *positive or affirma-*

tive quantities, or *additive* quantities; and those with the sign — are called *negative* or *subtractive* quanties. A quantity to which no sign is prefixed is understood to be *positive;* we need, therefore, prefix the positive sign only as a means of connecting the quantity to which it belongs to one which precedes. Thus, in the first example, below, the positive quantities $6ax$ and $7ax$ would be understood to be positive even if the + before them were omitted; the insertion of this sign is therefore in these cases superfluous; but in example 4, the positive quantity $4x$ requires the insertion of its sign to link it to the preceding quantity $6a$, which is itself positive, but has no such need for the sign.

ADDITION.

CASE 1.

(15.) ***When the quantities are like***, but ***have unlike signs.***

Add the coefficients of all the *positive* quantities into one sum, and those of the *negative* quantities into another.

Subtract the *less* sum from the *greater*.

Prefix the sign of the *greater* sum to the remainder, and annex the common letters.

The reason of this is evident: for the value of any number of quantities, taken collectively, of which some are to be *added*, and others to be *subtracted*, must be equal to the *difference* between all the *additive* quantities, and all the *subtractive* quantities.

EXAMPLES.

Add together the following quantities:

Ex. 1.	Ex. 2.	Ex. 3.
$+6ax$	$7xy$	$-4bx^2$
$-2ax$	$16xy$	$10bx^2$
$+7ax$	$-8xy$	$-7bx^2$
$-ax$	$2xy$	$-3bx^2$
Sum $10ax$	Sum $17xy$	Sum $-4bx^2$

4.

$$\begin{array}{rr} & 6a + 4x \\ & 4a + 8x \\ & -5a - 2x \\ & 7a - 3x \\ \hline \text{Sum} & 12a + 7x \\ \hline \end{array}$$

5.

$$\begin{array}{rr} & 2b + 8x \\ & -9b + 7x \\ & 4b + 2x \\ & 3b - 4x \\ \hline \text{Sum} & 13x \\ \hline \end{array}$$

6.

$$\begin{array}{r} 4a^2 + 6bx \\ 3a^2 + 5bx \\ 7a^2 - 4bx \\ 2a^2 + 2bx \\ \hline \\ \hline \end{array}$$

7.

$$\begin{array}{r} 7\sqrt{y} - 4(a+b) \\ 6\sqrt{y} + 2(a+b) \\ 2\sqrt{y} + (a+b) \\ \sqrt{y} - 3(a+b) \\ \hline \\ \hline \end{array}$$

8.

$$\begin{array}{r} a(a+b) + 3\sqrt{a-x} \\ -4a(a+b) + 7\sqrt{a-x} \\ 11a(a+b) - 6\sqrt{a-x} \\ -2a(a+b) - 2\sqrt{a-x} \\ 5a(a+b) + 14\sqrt{a-x} \\ \hline \\ \hline \end{array}$$

9.

$$\begin{array}{r} 7x^{\frac{1}{2}}y - 2x\sqrt{y} + 7 \\ x^{\frac{1}{2}}y + 3xy^{\frac{1}{2}} + 2 \\ 3x^{\frac{1}{2}}y - xy^{\frac{1}{2}} - 6 \\ 9x^{\frac{1}{2}}y - 4x\sqrt{y} - 3 \\ -2x^{\frac{1}{2}}y + 7x\sqrt{y} + 1 \\ \hline \\ \hline \end{array}$$

10.

$$\begin{array}{r} 4(x+y)^{\frac{1}{3}} + \sqrt{xyz} \\ -7(x+y)^{\frac{1}{3}} + 4\sqrt{xyz} \\ \sqrt{x+y} - 3(xyz)^{\frac{1}{3}} \\ -3(x+y)^{\frac{1}{3}} + 7(xyz)^{\frac{1}{3}} \\ -17(x+y)^{\frac{1}{3}} + 2\sqrt{xyz} \\ -3\sqrt{x+y} + (xyz)^{\frac{1}{3}} \\ \hline \\ \hline \end{array}$$

11.

$$\begin{array}{r} 5x\sqrt[3]{a+y} - 2x\sqrt[4]{y} + \sqrt{2} \\ 3x(a+y^{\frac{1}{3}} + 6xy^{\frac{1}{4}} + 2^{\frac{1}{3}} \\ -8x(a+y)^{\frac{1}{3}} - 4xy^{\frac{1}{2}} + 3\sqrt{2} \\ 7x\sqrt[3]{a+y} + 3x\sqrt[4]{y} + 2\sqrt{2} \\ 2x(a+y)^{\frac{1}{3}} + 5x\sqrt[4]{y} + 2^{\frac{1}{3}} \\ -9x\sqrt[3]{a+y} - 8xy^{\frac{1}{2}} - 8\sqrt{2} \\ \hline \\ \hline \end{array}$$

12.

$$
\begin{array}{l}
-3(ax+by+cz)^{\frac{1}{4}} - \sqrt{x^2+y^2} + a - b \\
2\sqrt[4]{ax+by+cz} + (x^2+y^2)^{\frac{1}{2}} - 3(a-b) \\
7\,\overline{ax+by+cz}^{\frac{1}{4}} - \sqrt{x^2+y^2} + 2(a-b) \\
3\sqrt[4]{(ax+by+cz)} + (x^2+y^2)^{\frac{1}{2}} + a - b \\
-5\sqrt{(ax+by+cz)} + (x^2+y^2)^{\frac{1}{2}} - 2\,\overline{a-b} \\
(ax+by+cz)^{\frac{1}{4}} - \sqrt{x^2+y^2} - 3\,\overline{a-b} \\
\hline
\\
\hline
\end{array}
$$

CASE 2.

(16.) ***When both quantities and signs are unlike, or some like and others unlike.***

Find the value of the *like* quantities, as in the preceding case, and connect to this value, by their proper signs, the *unlike* quantities.

Thus, in the first of the following examples, we find that there are *four* quantities like $x^{\frac{1}{2}}$, viz. *two* in the *first* column, and *two* in the *second*,* whose value, by the former case, is $2x^{\frac{1}{2}}$; also, there are *three* quantities like ax, one in each column, whose value is $3ax$; there are, likewise, *three* quantities like ab, whose value is $-4ab$; and there are *four* quantities like xy, whose value is $4xy$; but, as x has no *like*, it is merely connected to the value of the *like* quantities by its *sign* —.

1.

$$
\begin{array}{rl}
 & x^{\frac{1}{2}} + ax - ab \\
 & ab - \sqrt{x} + xy \\
 & ax + xy - 4ab \\
 & x^{\frac{1}{2}} + \sqrt{x} - x \\
 & xy + xy + ax \\
\hline
\text{Sum} & 2x^{\frac{1}{2}} + 3ax - 4ab + 4xy - x
\end{array}
$$

* The student must not forget that $x^{\frac{1}{2}}$ and $\sqrt{x}$ are *like*, each expression representing the *square root* of x, (Def. 6, page 12.)

2.

$$
\begin{array}{r}
3(c+d)x^3 + 4y - 2\sqrt{y} \\
6x^2y - 2ax + 12 \\
y + \sqrt{y} - 5ax \\
3ax - x^2y + (c+d)x^3 \\
x^2y + 2y - 14 \\
\hline
\text{Sum } 4(c+d)x^3 + 7y - \sqrt{y} + 6x^2y - 4ax - 2 \\
\hline
\end{array}
$$

3.

$$
\begin{array}{l}
\sqrt{x} + 3ax - 2\sqrt{b-x} \\
\frac{1}{2}ax - 2xy + 3\sqrt{x} \\
4xy + 3ax + (b-x)^{\frac{1}{2}} \\
\frac{1}{2}x^{\frac{1}{2}} + 8xy - 26 \\
7 - \sqrt{x} + ax \\
\hline
\\
\hline
\end{array}
$$

4.

$$
\begin{array}{r}
2ab + 12 - x^2y \\
x^{\frac{1}{2}}y + xy + 10 \\
3xy^{\frac{1}{2}} + 2x^2y - xy \\
5xy + 11 + x\sqrt{y} \\
17 - 2x^2y - x^{\frac{1}{2}}y \\
\hline
\\
\hline
\end{array}
$$

5.

$$
\begin{array}{r}
\sqrt{x^2+y^2} - \sqrt{x^2-y^2} + 2xy \\
(x^2-y^2)^{\frac{1}{2}} + \sqrt{x^2+y^2} - 3xy \\
4xy - (x^2-y^2)^{\frac{1}{2}} + \sqrt{x^2+y^2} \\
3(x^2+y^2)^{\frac{1}{2}} - 5xy + 7(x^2-y^2)^{\frac{1}{2}} \\
-7xy + 2\sqrt{x^2+y^2} - 3\sqrt{x^2-y^2} \\
\hline
\\
\hline
\end{array}
$$

6.

$$
\begin{array}{r}
2\sqrt[4]{xy} - 3a\sqrt{x} + x^2 - 13 \\
a\sqrt{x} + 12x^2 - 17 + (xy)^{\frac{1}{4}} \\
3x^2 - \sqrt{xy} + ax^{\frac{1}{2}} - 3 \\
-8(xy)^{\frac{1}{4}} + 9 - 2a\sqrt{x} + 3x^2 \\
x^2 + 3y^2 + 4\sqrt[4]{xy} - a\sqrt{x} \\
\hline
\\
\hline
\end{array}
$$

(17.) When the coefficients are *literal* instead of *numeral*, they are to be collected in a similar way; but here their several sums will have a *compound* form, as in the following examples:

1

$$\begin{array}{r} ax + by^2 \\ cdx + ady^2 \\ bx - cy^2 \\ \hline (a + cd + b)\,x + (b + ad - c)\,y^2 \\ \hline \end{array}$$

2.

$$\begin{array}{r} ax + dy^2 \\ by - dx \\ -by^2 + my \\ \hline (a - d)\,x + (d - b)\,y^2 + (b + m)\,y \\ \hline \end{array}$$

3.

$$\begin{array}{l} x^2 + adz \\ \frac{1}{2}x^2 - nz \\ bx^2 + cez \\ dx^2 - mz \\ \hline \\ \hline \end{array}$$

4.

$$\begin{array}{c} \sqrt{x} + by \\ ax - z \\ amy + c\sqrt{x} \\ dz + y \\ \hline \\ \hline \end{array}$$

5.

$$\begin{array}{c} a\sqrt{x^2 - y^2} + b\sqrt{x^2 + y^2} \\ c\sqrt{x^2 + y^2} - d\sqrt{x^2 - y^2} \\ f(x^2 + y^2)^{\frac{1}{2}} - e\,(x^2 - y^2)^{\frac{1}{2}} \\ 2\sqrt{x^2 + y^2} + 4a\sqrt{x^2 - y^2} \\ \sqrt{x^2 - y^2} - (x^2 + y^2)^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

6.

$$\begin{array}{r} (a + b)\sqrt{x} - (2 + m)\sqrt{y} \\ 4y^{\frac{1}{2}} + (a + c)\,x^{\frac{1}{2}} \\ 3n\sqrt{y} - (2d - e)\,x^{\frac{1}{2}} \\ (m + n)\,y^{\frac{1}{2}} + (b + 2c)\sqrt{x} \\ -2n\sqrt{x} + 12a\sqrt{y} \\ \hline \\ \hline \end{array}$$

SUBTRACTION.

(18.) Place the quantities to be subtracted underneath those they are to be taken from, as in arithmetic. Then conceive the signs of the quantities in the *lower* line to be changed from + to —, and from — to +, and collect the quantities together as if it were addition.

For if a positive quantity, as b, is to be taken from another quantity, as a, the difference will be represented by $a - b$, which is obviously the same as the addition of a and $-b$; but if $b - c$ is to be subtracted from a, then, since b is greater than $b - c$ by c, if b be subtracted, too much will be taken away by c; consequently, c must be added to the remainder to make up the deficiency; therefore, the true remainder is equal to the addition of $-b$ and c; that is, it is equal, as in the former instance, to the *addition* of the quantity to be subtracted with its *sign changed.*

EXAMPLES.

1.

From $5xy + 2x^{\frac{1}{2}} - 7a$

Take $3xy - x^{\frac{1}{2}} + 2a$

Remainder $2xy + 3x^{\frac{1}{2}} - 9a$

2.

From $\sqrt{x + y} + 3ax - 12$

Take $4\sqrt{x + y} - 2ax + b$

Rem. $-3\sqrt{x + y} + 5ax - 12 - b$

3.

From $3a\,(a - y) + 4by + a^3$

Take $2a\,(a - y) - 7by + 4a^3$

4.

$6x^2 + (x + y)^{\frac{1}{2}} - 10c$

$8x^2 - \sqrt{x + y} + 1$

5.

From $6abx + 12 - 3xy + 4xz$

Take $-3abx + xz - 7 + 5xy$

6.

From $\sqrt{x^2 - y^2} - 2(a + x)^{\frac{1}{2}} + 3$
Take $-3\sqrt{a + x} + 4(x^2 - y^2)^{\frac{1}{2}} - 1$

7.

From $2x(x + y)^{\frac{1}{4}} - 3axy + 2abc$
Take $-17axy + 11abc - x\sqrt[4]{x + y}$

Examples of quantities with literal coefficients.

1.

From $ax^2 + mxy + nx + b$
Take $sx^2 - pxy + qx - c$

$(a - s)x^2 + (m + p)xy + (n - q)x + b + c$

2.

From $pxy + qxz - rz^2 + s$
Take $mxy - pqxz - nz^2 + a$

3.

From $a(x - y)^{\frac{1}{2}} + bxy + c(a + x)^2$
Take $(x - y)^{\frac{1}{2}} - bxy + (a + c)(a + x)^2$

4.

From $(a + b)(x + y) - (c + d)(x - y) + m$
Take $(a - b)(x + y) + (c - d)(x - y) - n$

5.

$$\begin{array}{ll} \text{From} & (a - b)\,xy - (p + q)\,\sqrt{x + y} - hx^2 \\ \text{Take} & (2p - 3q)\,(x + y)^{\frac{1}{2}} - axy - (3 + h)\,x^2 \\ \hline \\ \hline \end{array}$$

MULTIPLICATION.

CASE 1.

(19.) ***When both multiplicand and multiplier are simple quantities.***

To the product of the coefficients annex the product of the letters, and it will be the whole product.

Thus, if it be required to multiply $6ax$ by $4b$, we have 24 for the product of the coefficients, and abx for the product of the letters; consequently, $24abx$ is the whole product; that is, $6ax \times 4b = 24abx$.

NOTE.—It must be particularly observed that quantities with *like* signs multiplied together, furnish a *positive* product whether the like signs be both $+$ or both $-$; and that quantities with *unlike* signs furnish a *negative* product. This may be expressed in short by the precept that *like signs multiplied together produce* plus, *and unlike signs* minus. The truth of this may be shown as follows:

1. Suppose any *positive* quantity b, is to be multiplied by any other *positive* quantity a; then b is to be taken as many times as there are units in a, and, as the sum of any number of positive quantities must be positive, the sign of the product ab must be $+$.

2. Suppose now that one factor b is negative, and the other a positive: then, as before, the product of $-b$ by a will be as many times $-b$ as there are units in a; and, since the sum of any number of negative quantities must be negative, the product in this case must be $-ab$.

3. If this last case be admitted, it will immediately follow that the product of $-b$ and $-a$ must be $+ab$, for if this be denied, the pro-

duct must be $-ab$, so that $-b$ multiplied by $+a$ produces the same as $-b$ multiplied by $-a$, which leads to the absurdity that $+a$ is the same thing as $-a$.

NOTE.—If *powers* of the same quantity are to be multiplied together, the operation is performed by simply *adding* the indices: thus, $a^2 \times a^3 = a^5$, for $a^2 = aa$, and $a^3 = aaa$, therefore $a^2 \times a^3 = aa \times aaa = aaaaa$, or a^5: also, $a^m \times a^n = a^{m+n}$, for $a^m = a \times a \times a \ldots .$ to m factors, and $a^n = a \times a \times a$ to n factors, and therefore $a^m \times a^n = (aaa \ldots .$ to m factors$) \times (aaa \ldots .$ to n factors), or (leaving out the sign $\times$) $= aaa \ldots .$ to $m+n$ factors $= a^{m+n}$. From this it follows, that in the *division* of powers the indices are to be *subtracted*.*

EXAMPLES.†

1. Multiply $6\sqrt{ax}$ by $4b$.

$$\text{Here } 6\sqrt{ax} \times 4b = 24b\sqrt{ax}.$$

2. Multiply $3x^2y^2$ by $2ax$.

$$3x^2y^2 \times 2ax = 6ax^3y^2.$$

3. Multiply $12x^4y$ by $-4a$.
4. Multiply $-4x^3y^2$ by $-4x^2y^3$.
5. Multiply $6axy^2$ by $3a^2bx^2$.
6. Multiply $13a^3b^2xy^4$ by $-8abx^3y^2$.
7. Multiply $\frac{1}{2}x^2y^3z^4$ by $6x^4y^3z^2$.
8. Multiply $-9cxy^2z^5$ by $-\frac{1}{3}c^2x^2y^2z^2$.

---

* This mode of proof does not apply when the quantities to be multiplied have fractional indices, although the rule still holds. Thus, let the product of $a^{\frac{1}{2}}$ and $a^{\frac{3}{5}}$ be required, then the exponents $\frac{1}{2}$, $\frac{3}{5}$, in a common denominator, are $\frac{5}{10}$, $\frac{6}{10}$; hence the proposed factors are the same as $a^{\frac{5}{10}}$ and $a^{\frac{6}{10}}$; that is, the 5th power of the 10th root of a, and the 6th power of the same root; we have therefore, as above,

$$\left(a^{\frac{1}{10}}\right)^5 \times \left(a^{\frac{1}{10}}\right)^6 = \left(a^{\frac{1}{10}}\right)^{11} = a^{\frac{11}{10}}.$$

† Although the product of the letters will be the same in value in whatever order we arrange them, yet in these examples the student, conformably to the usual custom, is expected to arrange them according to their order in the alphabet.

CASE II.

(20.) ***When the multiplicand is a compound quantity, and the multiplier a simple quantity.***

Find the product of the multiplier and each term of the multiplicand *separately*, beginning at the left hand; connect these products by their proper signs, and the complete product will be exhibited.

EXAMPLES.

1. Multiply $ax + b$ by $4x^2$

$$\begin{array}{l} ax \quad + b \\ 4x^2 \\ \hline 4ax^3 + 4bx^2 \\ \hline \end{array}$$

2. Multiply $12xy - ax + 6$ by $3xy$.

$$\begin{array}{l} 12xy - ax + 6 \\ \ 3xy \\ \hline 36x^2y^2 - 3ax^2y + 18xy \\ \hline \end{array}$$

3. Multiply $5ab + 3a - 2$ by $5xy$.
4. Multiply $31xy^2 - 4\sqrt{x} + a$ by $-2\sqrt{b}$.
5. Multiply $12x^3y + 2xy^2 + xy$ by $3ax$.
6. Multiply $4abx + 3cy - abc$ by $3xy^2$.
7. Multiply $3x^2y^3 - 4xy^2 + bx$ by $-7x^3y$.
8. Multiply $-5axy^2 + \frac{1}{2}x^2 - \frac{1}{4}ay^3$ by $8axy$.
9. Multiply $\frac{1}{3}\sqrt{z} - \frac{2}{3}ax^2 - \frac{1}{2}xy^2$ by $-6a^2x^2$.

CASE III.

(21.) ***When both multiplicand and multiplier are compound quantities.***

Multiply each term in the multiplier by all the terms in the multiplicand.

Connect the several products by their proper signs, as in the last case, and their sum will be the whole product.

EXAMPLES.

1. Multiply $a+b$ by $a+b$.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \qquad ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array}$$

2. Multiply $a+b$ by $a-b$.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \\ \hline \end{array}$$

3. Multiply $x+\frac{1}{2}y-2$ by $\frac{1}{4}x+3y$.

$$\begin{array}{l} x+\frac{1}{2}y-2 \\ \frac{1}{4}x+3y \\ \hline \frac{1}{4}x^2+\frac{1}{8}xy-\frac{1}{2}x \\ \qquad 3xy+\frac{3}{2}y^2-6y \\ \hline \frac{1}{4}x^2+3\frac{1}{8}xy-\frac{1}{2}x+\frac{3}{2}y^2-6y \\ \hline \end{array}$$

4. Find the four first terms in the product of

$a^m+a^{m-1}x+a^{m-2}x^2+a^{m-3}x^3+$ &c. and $a+x$.

$$\begin{array}{l} a^m+a^{m-1}x+a^{m-2}x^2+a^{m-3}x^3+\&c. \\ a+x \\ \hline a^{m+1}+a^mx+a^{m-1}x^2+a^{m-2}x^3+\&c. \\ \qquad\quad a^mx+a^{m-1}x^2+a^{m-2}x^3+\&c. \\ \hline a^{m+1}+2a^mx+2a^{m-1}x^2+2a^{m-2}x^3+\&c. \\ \hline \end{array}$$

5. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.

Ans. $x^4 \quad y^4$

6. Multiply a^n+b^n by $a-b$.

Ans. $a^{n+1}+ab^n-a^nb-b^{n+1}$.

7. Multiply $a+x+x^2+x^3+x^4$ by $a-x$.

Ans. $a^2+(a-1)x^2+(a-1)x^3+(a-1)x^4-x^5$.

8. Multiply $x^4-x^3+x^2-x+1$ by x^2+x-1.

Ans. $x^6-x^4+x^3-x^2+2x-1$.

9. Multiply $3x^2+(x+y)^{\frac{1}{2}}-7$ by $2x^2+\sqrt{x+y}$.

Ans. $6x^4+(5x^2-7)\sqrt{x+y}-14x^2+x+y$.

10. Multiply $ax+bx^2+cx^3$ by $1+x+x^2+x^3$.

Ans. $ax+\begin{array}{c|}a\\b\\ \end{array}x^2+\begin{array}{c|}a\\b\\c\end{array}x^3+\begin{array}{c|}a\\b\\c\end{array}x^4+\begin{array}{c|}b\\c\\ \end{array}x^5+cx^6$.

DIVISION.

CASE 1.

(22.) *When both dividend and divisor are simple quantities.*

To the quotient of the *coefficients* annex the quotient of the *letters*,* and it will be the whole quotient.

NOTE. The rule for the signs must be observed here, as well as in multiplication.

EXAMPLES.

1. Divide $12ax$ by $3a$.

$$\frac{12ax}{3a}=4x.$$

2. Divide $24x^2y$ by $3xy$.

$$\frac{24x^2y}{3xy}=8x.$$

* The learner will readily discover the quotient of the *letters* by asking himself, *what letters must I join to those in the divisor to make those in the dividend?* Thus in ex. 3, next page, the dividend contains two x's, two y's, and two z's, while the divisor contains but one x and one z; so that to make up the letters in the dividend, I must join to those in the divisor one x, two y's, and one z; that is, the quotient of the letters will be xy^2z.

3. Divide $-16x^2y^2z^2$ by $-4xz$.

$$\frac{-16x^2y^2z^2}{-4xz}=4xy^2z.$$

4. Divide $9a^3x^4$ by $3ax^2$.
5. Divide $26ax^2y^2$ by $-2xy$.
6. Divide $-15b^3xy^5$ by $-bxy^2$.
7. Divide $28c^4z^6$ by $-7c^3z^6$.
8. Divide $-18a^2b^3y^7z^4$ by $-aby^3z$.

CASE 2.

23. ***When the dividend is a compound quantity, and the divisor a simple quantity.***

Find the quotient of the divisor and each term of the dividend ***separately***, connect these quotients together by their proper signs, and the whole quotient will be exhibited.

EXAMPLES.

1. Divide $12a^2x+4ax^2-16a$ by $4a$.

$$\frac{12a^2x+4ax^2-16a}{4a}=3ax+x^2-4.$$

2. Divide $a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x$ by a^n.

$$\frac{a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x}{a^n}=ax-a^2x-a^3x-a^4x.$$

3. Divide $24a^2x^4+6a^3x^3-3a^4x^2+12ax$ by $-3ax$.
4. Divide $ax^n+ax^{n+1}+ax^{n+2}+ax^{n+3}+$ &c. by x^n.
5. Divide $6(x+y)^3-8(x+y)^2+4a^2(x+y)$ by $2(x+y)$.
6. Divide $ax^{m-1}+bx^{m+1}-cx^{m-3}+dx^5$ by x^{m-6}.

CASE 3.

(24.) ***When both dividend and divisor are compound quantities.***

Arrange both dividend and divisor according to the ***powers of some letter*** common to both; that is, let the ***first*** term, in both dividend and divisor, be that which contains the ***highest power*** of the same letter, the second the ***next*** highest, and so on.

Find how often the first term in the divisor is contained in that of the dividend, and it will give the first term in the quotient, by which all the terms in the divisor must be multiplied, and the product subtracted from the dividend.

To the remainder annex as many of the other terms of the dividend as are found requisite, and proceed to find the next term in the quotient, as in common arithmetic.

EXAMPLES.

1. Divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

$$\begin{array}{l}
x^2 + x - 1)\, x^6 - x^4 + x^3 - x^2 + 2x - 1\, (x^4 - x^3 + x^2 - x + 1 \\
\qquad\qquad x^6 + x^5 - x^4 \\
\hline
\qquad\qquad\quad - x^5 + x^3 - x^2 \\
\qquad\qquad\quad - x^5 - x^4 + x^3 \\
\hline
\qquad\qquad\qquad\qquad x^4 - x^2 + {}^2x \\
\qquad\qquad\qquad\qquad x^4 + x^3 - x^2 \\
\hline
\qquad\qquad\qquad\qquad\quad - x^3 + 2x - 1 \\
\qquad\qquad\qquad\qquad\quad - x^3 - x^2 + x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad x^2 + x - 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad x^2 + x - 1 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad * \quad * \quad * \\
\hline
\end{array}$$

2. Divide $a^n - x^n$ by $a - x$.

$$\begin{array}{l}
a - x)\, a^n - x^n\, (a^{n-1} + a^{n-2}x + a^{n-3}x^2 + a^{n-4}x^3 + \text{\&c.} \\
\qquad\quad a^n - a^{n-1}x \\
\hline
\qquad\qquad\quad a^{n-1}x - x^n \\
\qquad\qquad\quad a^{n-1}x - a^{n-2}x^2 \\
\hline
\qquad\qquad\qquad\qquad a^{n-2}x^2 - x^n \\
\qquad\qquad\qquad\qquad a^{n-2}x^2 - a^{n-3}x^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad a^{n-3}x^3 - x^n \\
\qquad\qquad\qquad\qquad\qquad\quad a^{n-3}x^3 - a^{n-4}x^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad a^{n-4}x - x^{4n} \text{ \&c.}
\end{array}$$

3. Divide $1 + ax + bx^2 + cx^3 + dx^4 +$ &c. by $1 - x$.

$$1 - x)\ 1 + ax + bx^2 + cx^3 + dx^4\ \Big(1 + \left.\begin{array}{r}1\\a\end{array}\right| x + \left.\begin{array}{r}1\\a\\b\end{array}\right| x^2 + \left.\begin{array}{r}1\\a\\b\\c\end{array}\right| x^3 + \text{\&c.}$$

$$\underline{1 - x}$$

$$\left.\begin{array}{r}1\\a\end{array}\right| x + bx^2$$

$$\underline{\left.\begin{array}{r}1\\a\end{array}\right| x \left.\begin{array}{r}-1\\-a\end{array}\right| x^2}$$

$$\left.\begin{array}{r}1\\a\\b\end{array}\right| x^2 + cx^3$$

$$\underline{\left.\begin{array}{r}1\\a\\b\end{array}\right| x^2 \left.\begin{array}{r}-1\\-a\\-b\end{array}\right| x^3}$$

$$\left.\begin{array}{r}1\\a\\b\\c\end{array}\right| x^3 + dx^4$$

$$\underline{\left.\begin{array}{r}1\\a\\b\\c\end{array}\right| x^3 \left.\begin{array}{r}-1\\-a\\-b\\-c\end{array}\right| x^4} \quad \text{\&c. \&c.}$$

4. Divide $a^5 + x^5$ by $a + x$.

Ans. $a^4 - a^3x + a^2x^2 - ax^3 + x^4$.

5. Divide $a^5 - x^5$ by $a - x$.

Ans. $a^4 + a^3x + a^2x^2 + ax^3 + x^4$.

6. Divide $x^3 + \frac{3}{4}x^2 + \frac{3}{4}x + 1$ by $\frac{1}{2}x + \frac{1}{2}$.

Ans. $2x^2 - \frac{1}{2}x + 2$.

7. Divide 1 by $1 - x$.

Ans. $1 + x + x^2 + x^3 + x^4 + x^5 +$ &c.

8. Divide $x^4 - y^4$ by $x^3 + x^2y + xy^2 + y^3$.

Ans. $x - y$.

9. Divide $y^{m+1} + yx^m - y^mx - x^{m+1}$ by $y^m + x^m$.

Ans. $y - x$.

10. Divide $a - bx + cx^2 - dx^3 +$ &c. by $1 + x$.

Ans. $a \left.\begin{array}{r}-a\\-b\end{array}\right| x \left.\begin{array}{r}+a\\+b\\+c\end{array}\right| x^2 - \text{\&c.}$

SCHOLIUM.

From the preceding rules are deduced the following useful theorems, viz.

1. By the rule for addition, if the sum of any two quantities, a and b, be added to their difference, the sum will be twice the greater.*

2. By the rule for subtraction, if the difference of any two quantities be taken from their sum, the remainder will be twice the less.†

3. By multiplication, Article 21, Example 2, page 24, if the sum of any two quantities be multiplied by their difference, the product will be the difference of their squares.

ALGEBRAIC FRACTIONS.

THE operations performed on ALGEBRAIC FRACTIONS are similar to those performed on *numeral* fractions in Arithmetic, and which are as follow:

To reduce a Mixed Quantity to an Improper Fraction.

(25.) Multiply the quantity to which the fraction is annexed, by the denominator of the fraction; connect the product, by the proper sign, to the numerator; place the denominator underneath, and we have the improper fraction required.

Thus, if it were required to reduce $ab - \frac{c}{d}$ to an improper fraction, then ab, the quantity to which the fraction is annexed, multiplied by d, the denominator, gives abd; which annexed to the numerator, c, by the proper sign $-$, gives $abd - c$; therefore the improper fraction is $\frac{abd - c}{d}$.

* For $\left\{\begin{array}{r} \ldots\ldots a+b \\ \text{added to } a-b \\ \hline \text{gives } 2a \end{array}\right.$

† and $\left\{\begin{array}{r} \ldots\ldots\ldots a+b \\ \text{diminished by } a-b \\ \hline \text{gives } 2b \end{array}\right.$

EXAMPLES.

1. Reduce $(a+b)+\frac{ax}{y}$ to an improper fraction.

$$(a+b)+\frac{ax}{y}=\frac{(a+b)y+ax}{y}.$$

2. Reduce $3ax-\frac{a-b}{y}$ to an improper fraction.

$$3ax-\frac{a-b}{y}=\frac{3axy-(a-b)}{y}.$$

Here the expression $-(a-b)$ signifies that $a-b$ is to be subtracted from that which precedes, and therefore the signs of a and b must be *changed*, (Art. 18, page 19:) consequently,

$$\frac{3axy-(a-b)}{y}=\frac{3axy-a+b}{y};$$

the same must be observed in the following, and in every similar example.

3. Reduce $4x-\frac{3x-b+4}{10}$ to an improper fraction.

$$4x-\frac{3x-b+4}{10}=\frac{37x+b-4}{10}$$

4. Reduce $2ay+\frac{4x-ay+2}{2xy}$ to an improper fraction.

Ans. $\frac{4axy^2+4x-ay+2}{2xy}$.

5. Reduce $a-x-\frac{a^2-ax}{x}$ to an improper fraction.

Ans. $\frac{2ax-x^2-a^2}{x}$.

6. Reduce $ax-y-\frac{3ax-4y-2}{5}$ to an improper fraction.

Ans. $\frac{2ax-y+2}{5}$.

7. Reduce $a+b-\frac{a^2-b^2-3}{a-b}$ to an improper fraction.

Ans. $\frac{3}{a-b}$.

8. Reduce $x^4-x^3y+x^2y^2-xy^3+y^4-\dfrac{1}{x+y}$ to an improper fraction.

Ans. $\dfrac{x^5+y^5-1}{x+y}$.

To reduce an Improper Fraction to a Whole or Mixed Quantity.

(26.) Divide the numerator by the denominator, and, if there be a remainder, place it under the denominator; connect this fraction, by its proper sign, to the quotient, and we shall have the mixed quantity required.

Thus, if it were proposed to reduce $\dfrac{abd-c}{d}$ to a mixed quantity, we have only to perform the actual division of the numerator by the denominator, and we get for the *quotient* ab, and for the *remainder* $-c$; therefore $ab-\dfrac{c}{d}$ is the improper fraction required.

EXAMPLES.

1. Reduce $\dfrac{4x^2+ax-2}{2x}$ to a mixed quantity.

$$\frac{4x^2+ax-2}{2x}=2x+\frac{ax-2}{2x}.$$

2. Reduce $\dfrac{2xy-a}{xy}$ to a mixed quantity. Ans. $2-\dfrac{a}{xy}$.

3. Reduce $\dfrac{x^2-y^2+4}{x+y}$ to a mixed quantity. Ans. $x-y+\dfrac{4}{x+y}$.

4. Reduce $\dfrac{3(a^5+b^5)-3}{a+b}$ to a mixed quantity.

Ans. $3a^4-3a^3b+3a^2b^2-3ab^3+3b^4-\dfrac{3}{a+b}$.

5. Reduce $\dfrac{4axy^2+4x-ay+2}{2xy}$ to a mixed quantity.

Ans. $2ay+\dfrac{4x-ay+2}{2xy}$.

6. Reduce $\dfrac{x^3-y^3+x^2-2y^2}{x-y}$ to a mixed quantity.

Ans. $x^2+xy+y^2+x+y-\dfrac{y^2}{x-y}$.

To find the greates Common Measure of the Terms of a Fraction.

(27.) Arrange the numerator and denominator according to the powers of some letter, as in division, making that the *dividend* which divisor contains the *highest* power, and the other the divisor.

Perform the division, and consider the *remainder* as a new *divisor*, and the last divisor a new dividend; then consider the remainder that arises from this division as another new divisor, and the last divisor the corresponding dividend. Continue this process till the remainder is 0, and the last divisor will be the greatest common measure sought.

Note. If any quantity be common to all the terms of either of the divisors, but *not* common to those of the corresponding dividend, this quantity may be expunged from the divisor.

The truth of the above process depends chiefly upon the two following properties:

1. If a quantity divide another, it will also divide any multiple of it: If, for instance c divide b, and the quotient be n, it will also divide rb, and the quotient will be rn.

2. If a quantity divide each of two others, it will also divide their sum and difference: For, let c divide a, and call the quotient m; let it also divide b, and call the quotient n, then $a = mc$, and $b = nc$; therefore $a \mp b$* $= mc \mp nc$; now c evidently measures $mc \mp nc$, consequently it measures its equal, $a \mp b$.

(28.) Let now $\frac{a}{b}$ represent any fraction, and let the work in the margin be carried on according to the rule (Art. 27), c being put for the *first* remainder, d for the *second*, &c. Then $a - br = c$, and, if c divide b, c will be the *last* divisor, and the work will be finished, d being then $= 0$: hence, in this case, since c divides b, it also divides br; and since it divides $a - br$ (this being equal to c), it must also divide a; c therefore divides both a and b. It is moreover evident that c must be the *greatest* common divisor of a and br; for, if these quantities had a divisor *greater* than c, then

$$\begin{array}{l} b)\,a\,(r \\ \quad br \\ \quad \overline{} \\ \quad c)\,b\,(s \\ \qquad cs \\ \qquad \overline{} \\ \qquad d)\,c\,(t \\ \qquad\quad dt \\ \qquad\quad \overline{} \\ \qquad\quad 0 \\ \qquad\quad \overline{} \end{array}$$

* The double sign $\mp$ signifies *plus* or *minus*.

their difference $a - br = c$, would be divisible by it, as has been proved above; that is, c would be divisible by a quantity ***greater than itself***, which is absurd; c, therefore, is the ***greatest*** common divisor of a and br, and, consequently, of a and b. Suppose, however, that the work does not end here, and that the last divisor is d, then $b - cs = d$; and since d divides c, it also divides cs, and consequently b; d therefore divides both b and c, and must consequently divide $br + c$, or a; and, since d is the ***greatest*** divisor of $b - cs$, it must necessarily be the ***greatest*** common divisor of b and cs, and therefore of b and c; whence d is the greatest common divisor of a, b, and c, and consequently of a and b, since whatever divides a and b must also divide $a - br$, or its equal c. The reasoning will be similar, whatever be the length of the operation.*

With reference to the note, it may be observed, that, by expunging any quantity common to all the terms of a divisor, we do not destroy any ***common measure*** of that divisor and its corresponding dividend, since no factor of the quantity expunged is supposed to exist in ***all*** the terms of the dividend.

EXAMPLES.

1. Find the greatest common measure of the terms of the fraction,

$$\frac{a^4 - x^4}{a^3 + a^2x - ax^2 - x^3}.$$

* The method of finding the greatest common measure of any two quantities may be easily extended to the finding the greatest common measure of three or more quantities. For, let a, b, c, represent any three quantities, and let x be the greatest common measure of a and b, and y the greatest common measure of c and x; then, since whatever measures x, measures also a and b; whatever measures c and x, measures also a, b, c; therefore the greatest common measure of c and x is also the greatest common measure of a, b, and c; $\therefore$ y is the greatest common measure. If, again, z be the greatest common measure of y and d, then will z be also the greatest common measure of a, b, c, and d, &c. The chief use of the greatest common measure of the terms of a fraction, is to reduce the fraction to its simplest form. In many fractions this common measure is discernible at sight, these therefore may be simplified without the aid of the above rule.

Arranging the terms according to the powers of a,

$$
\begin{array}{l}
a^3+a^2x-ax^2-x^3)\ a^4-x^4\ (a-x \\
\qquad\qquad\qquad\quad \underline{a^4+a^3x-a^2x^2-ax^3} \\
\qquad\qquad\qquad\qquad -a^3x+a^2x^2+ax^3-x \\
\qquad\qquad\qquad\qquad \underline{-a^3x-a^2x^2+ax^3+x^4} \\
\qquad\qquad\qquad\qquad\qquad\quad 2a^2x^2-2x^4
\end{array}
$$

$$
\begin{array}{r}
2a^2x^2-2x^4 \\
\text{or } \textit{expunging } 2x^2 \\
a^2-x^2
\end{array}
\left)
\begin{array}{l}
\\
a^3+a^2x-ax^2-x^3\ (a+x \\
\underline{a^3-ax^2} \\
\quad a^2x-x^3 \\
\quad \underline{a^2x-x^3} \\
\quad \underline{\ *\qquad *\ }
\end{array}
\right.
$$

Whence it appears that a^2-x^2 is the greatest common measure of the terms of the proposed fraction, and, consequently, by dividing both numerator and denominator by this common measure, the fraction is reduced to its lowest terms, and becomes $\dfrac{a^2+x^2}{a+x}$.

2. It is required to reduce $\dfrac{a^4-x^4}{a^5-a^3x^2}$ to its lowest terms.

Arranging the terms according to the powers of a;

$$
\begin{array}{c}
a^5-a^3x^2 \\
\text{or} \\
a^2-x^2
\end{array}
\left)
\begin{array}{l}
a^4-x^4\ (a^2+x^2 \\
\underline{a^4-a^2x^2} \\
a^2x^2-x^4 \\
\underline{a^2x^2-x^4} \\
\underline{\ *\qquad *\ }
\end{array}
\right.
$$

The greatest common measure being a^2-x^2, the fraction in its lowest terms is $\dfrac{a^2+x^2}{a^3}$.

3. Reduce the fraction $\dfrac{2ax^2 - a^2x - a^3}{2x^2 + 3ax + a^2}$ to its most simple form.*

Ans. $\dfrac{ax - a^2}{x + a}$.

4. Reduce the fraction $\dfrac{6ax^3 + ax^2 - 12ax}{6ax - 8a}$ to its lowest terms.

Ans. $\dfrac{2x^2 + 3x}{2}$.

5. Reduce the fraction $\dfrac{x^4 - y^4}{x^3 + y^3}$ to its most simple form.

Ans. $\dfrac{x^3 - x^2y + xy^2 - y^3}{x^2 - xy + y^2}$.†

6. Reduce the fraction $\dfrac{3x^3 - 24x - 9}{2x^3 - 16x - 6}$ to its lowest terms. Ans. $\frac{3}{2}$.

(29.) From having the greatest common measure of two quantities, their least common multiple may be obtained, this being equal to the product of the two quantities divided by their greatest common measure: For, let x be the greatest common measure of a and b, and put $\dfrac{a}{x} = p$; and $\dfrac{b}{x} = q$; then p and q cannot have a common measure; now, since $a = px$, and $b = qx$, and since pq is the least common multiple of p and q, and therefore pqx the least of px and qx; pqx (or its equal $\dfrac{ab}{x}$), must be the least common multiple of their equals, a and b. The least common multiple of three quantities is had by first finding that of two, and then the least common multiple of it and the other quantity, &c.

To Reduce Fractions to a Common Denominator.

(30.) Multiply each numerator, separately, into all the denominators, except its own, and the products will be the new numerators.

Multiply all the denominators together, and the product will be the common denominator.

---

* In finding the common measure, either numerator or denominator may be taken as the first divisor, whichever is found most convenient.

† This fraction appears less simple than the original one, but it is in reality more so, the numerator and denominator of the former being, respectively, $x+y$ times that of the latter.

That this process alters the *form* merely, and not the *value* of the several fractions, will appear from observing that the numerator and denominator of each fraction are both multiplied by the same quantity, viz. by the product of the denominators of the other fractions.

NOTE. If one of the given denominators should happen to be equal to the product of all the others, (as in Example 1, following,) then this denominator will obviously be the same as the common denominator, found by the above rule, for the other fractions; so that it will be sufficient to operate upon these only in order to reduce the whole to a common denominator. (See the second solution to the first Example.)

EXAMPLES.

1. Reduce the fractions $\frac{a}{xy}$, $\frac{ax}{y}$, and $\frac{a}{x}$ to a common denominator.

$$\left.\begin{aligned} &axy \\ ax \times xy \times x &= ax^3y \\ a \times xy \times y &= axy^2 \end{aligned}\right\} \text{the new numerators.}$$

$$xy \times y \times x = x^2y^2 = \text{the common denominator};$$

$$\therefore \text{ the three fractions are } \frac{axy}{x^2y^2}, \frac{ax^3y}{x^2y^2}, \text{ and } \frac{axy^2}{x^2y^2}.$$

Since, however, in each of these fractions xy is common to both numerator and denominator, this quantity may be expunged, and the fractions written in the following more simple form:

$$\frac{a}{xy}, \frac{ax^2}{zy}, \frac{ay}{xy}.$$

But, by attending to the NOTE (above) we arrive at once at these simplified forms. Thus, taking the second and third fractions only, as the product of their denominators gives the denominator of the first, the process will be

$$\left.\begin{aligned} ax \times x &= ax^2 \\ a \times y &= ay \end{aligned}\right\} \text{the new numerators,}$$

$$xy \text{ the common denominator};$$

hence the three fractions are

$$\frac{a}{xy}, \frac{ax^2}{xy}, \frac{ay}{xy}.$$

2. Reduce the fractions $\frac{4}{ax}$, $\frac{2a}{x}$, and $\frac{3}{4}$, to a common denominator.

Ans. $\frac{16x}{4ax^2}$, $\frac{8a^2x}{4ax^2}$, and $\frac{3ax^2}{4ax^2}$

Or more simply, $\frac{16}{4ax}$, $\frac{8a^2}{4ax}$, and $\frac{3ax}{4ax}$.

3. Reduce $\frac{2x+1}{a}$ and $\frac{x+a}{3}$ to a common denominator.

Ans. $\frac{6x+3}{3a}$ and $\frac{ax+a^2}{3a}$.

4. Reduce $\frac{2x^3-a}{2a}$, a, and 4, to fractions having a common denominator.*

Ans. $\frac{2x^3-a}{2a}$, $\frac{2a^2}{2a}$, and $\frac{8a}{2a}$.

5. Reduce $\frac{3x^2-2}{4a}$, and $\frac{2x^2-x+4}{a+x}$ to a common denominator.

Ans. $\frac{3ax^2+3x^3-2x-2a}{4a^2+4ax}$ and $\frac{8ax^2-4ax+16a}{4a^2+4ax}$.

6. Reduce $\frac{a}{x+y}$, $\frac{b}{x-y}$, $\frac{c}{x^2-y^2}$, to a common denominator.†

Ans. $\frac{a(x-y)}{x^2-y^2}$, $\frac{b(x+y)}{x^2-y^2}$, $\frac{c}{x^2-y^2}$.

7. Reduce $\frac{a-x}{x}$, $\frac{a+x}{x(a^2-x^2)}$, $\frac{a-x}{a+x}$, $\frac{1}{a-x}$, to a common denominator.

Ans. $\frac{(a+x)(a-x)^2}{x(a^2-x^2)}$, $\frac{a+x}{x(a^2-x^2)}$, $\frac{x(a-x)^2}{x(a^2-x^2)}$, $\frac{x(a+x)}{x(a^2-x^2)}$

---

* Whole quantities may be put under a fractional form by making their denominators unity: thus,

$$a=\frac{a}{1} \text{ and } 4=\frac{4}{1}, \text{ \&c.}$$

† See Note, page 36. The student will have frequent occasion for the property mentioned at page 29, viz. that *the sum multiplied by the difference of two quantities gives the difference of their squares.*

ADDITION OF FRACTIONS.

(31.) Reduce the fractions to a common denominator. Add the numerators together, and under the sum place the common denominator.

EXAMPLES.

1. Add together $\frac{2b}{x^2+b^2}$, and $\frac{1}{x}$.

$$\frac{2b}{x^2+b^2}+\frac{1}{x}=\frac{2bx}{x^3+b^2x}+\frac{x^2+b^2}{x^3+b^2x}=\frac{x^2+2bx+b^2}{x^3+b^2x} \text{ the sum required.}$$

2. Add together $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. Ans. $\frac{2x^2+2y^2}{x^2-y^2}$.

3. Add together $\frac{3x+2}{a}$, $\frac{4x+3}{b}$, and $\frac{5x+4}{c}$.

Ans. $\frac{bc(3x+2)+ac(4x+3)+ab(5x+4)}{abc}$.

4. Add together $\frac{2a}{b}$, $\frac{3a^2}{6}$, $\frac{2b}{a}$, and $\frac{1}{2}$.

Ans. $\frac{4a^2+a^3b+4b^2+ab}{2ab}$.

5. Required the sum of $\frac{x}{x^2-y^2}$, $\frac{y}{x+y}$, and $\frac{1}{x-y}$. (See Note, page 36.)

Ans. $\frac{2x+xy-y^2+y}{x^2-y^2}$.

6. Express $\frac{p}{3my^2-x}+\frac{y-6mpy^2}{(3my^2-x)^2}$ in a single fraction.

Ans. $\frac{y-3mpy^2-px}{(3my^2-x)^2}$.

SUBTRACTION OF FRACTIONS.

(32.) Reduce the fractions to a common denominator, which place under the difference of the numerators.

EXAMPLES.

1. Subtract $\frac{12x}{a}$ from $\frac{6ax}{5}$.

$\frac{6ax}{5}-\frac{12x}{a}=\frac{6a^2x}{5a}-\frac{60x}{5a}=\frac{(6a^2-60)x}{5a}$ the difference required.

2. Subtract $\frac{2x+1}{3}$ from $\frac{7x}{2}$. Ans. $\frac{17x-2}{6}$.

3. Subtract $\frac{3x+2}{x-1}$ from $\frac{5x-3}{x+1}$. Ans. $\frac{2x^2-13x+1}{x^2-1}$.

4. Subtract $\frac{1}{x+y}$ from $\frac{1}{x-y}$. Ans. $\frac{2y}{x^2-y^2}$.

5. Subtract $\frac{1}{x^2-y^2}$ from $\frac{1}{x-y}$. Ans. $\frac{x+y-1}{x^2-y^2}$.

6. Subtract $\frac{2x^2-13x+1}{x^2-1}$ from $\frac{5x-3}{x+1}$. Ans. $\frac{3x+2}{x-1}$.

MULTIPLICATION OF FRACTIONS.

(33.) Multiply the numerators together, and it will give the numerator of the product.

Multiply the denominators together, and it will give the denominator of the product.

EXAMPLES.

1. Multiply $\frac{4ax}{3}$ by $\frac{2a}{5}$.

$$\frac{4ax}{3}\times\frac{2a}{5}=\frac{8a^2x}{15}\text{ the product required.}$$

2. Multiply $\frac{2x+3y}{a}$ by $\frac{2a}{x}$.

Ans. $\frac{4x+6y}{x}$ = the product in its lowest terms.

3. Multiply $\frac{a-x^2}{2}$ by $\frac{2a}{a-x}$. Ans. $\frac{a^2-ax^2}{a-x}$.

4. Multiply $\frac{a+x}{a}$, $\frac{a-x}{x}$, and $\frac{a^2-x^2}{a^2+x^2}$, together.

Ans. $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$.

5. Multiply $\frac{x^2-y^2}{x}$, $\frac{x}{x+y}$ and $\frac{1}{x-y}$ together. Ans. 1.

6. Multiply $\frac{3(a^2-x^2)+a-x}{2}$ by $\frac{4}{3(a-x)}$.

Ans. $\frac{6(a+x)+2}{3}$.

DIVISION OF FRACTIONS.

(34.) Divide by the numerator of the divisor, and multiply by the denominator; Or, which is the same thing, invert the divisor, and proceed as in multiplication.

Thus, if $\frac{x}{2}$ is to be divided by $\frac{2x}{7}$; then, dividing $\frac{x}{2}$ by $2x$, gives $\frac{x}{4x}$. But $2x$ is 7 times $\frac{2x}{7}$; therefore, as the divisor was 7 times too great, the quotient must be 7 times too little; consequently,

$$\frac{x}{4x} \times 7 = \frac{7x}{4x} = \frac{7}{4}.$$

is the true quotient.

EXAMPLES.

1. Divide $\frac{4x+6}{3}$ by $\frac{x+3}{2x}$.

$$\frac{4x+6}{3} \times \frac{2x}{x+3} = \frac{8x^2+12x}{3x+9} = \text{the quotient.}$$

2. Divide $\frac{ax+b}{a}$ by $\frac{bx-a}{b}$. Ans. $\frac{abx+b^2}{abx-a^2}$.

3. Divide $\frac{6(a+x)+2}{3}$ by $\frac{4}{3(a-x)}$.

Ans. $\frac{3(a^2-x^2)+(a-x)}{2}$.

4. Divide $a+\frac{2ax-1}{b}$ by $\frac{x-a}{ax+1}$.

Ans. $\frac{a^2(bx+2x^2)+a(x+b)-1}{b(x-a)}$.

5. Divide 12 by $\frac{(a+x)^2}{x}-a$. Ans. $\frac{12x}{a^2+ax+x^2}$.

6. Divide $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$ by $\frac{a^2-x^2}{a^2+x^2}$. Ans. $\frac{a^2-x^2}{ax}$.

INVOLUTION.

(35.) Involution is the raising of quantities to any proposed power.

If the quantity to be involved be a single letter, the involution is represented by placing the number of the power a little above it, as was observed in the definitions at the beginning.

The power of a simple quantity, consisting of more than one letter, is also similarly represented: Thus, the square or second power of abc is $(abc)^2$, or $\overline{abc}^{\,2}$, or $a^2b^2c^2$, the third power is $(abc)^3$, &c.

(36.) If the simple quantity be some power already, or if it be composed of factors that are powers, then the index, or indices, must be multiplied by the index of the power to which the quantity is to be raised: Thus, the second power of a^3 is a^6, because $a^3\times a^3=a^6$; also, the nth power of a^3 is a^{3n}. because $a^3\times a^3\times a^3\times a^3$ . . . . . to n factors is a^{3n}; the nth power of a^m is a^{mn}, because, in

like manner, $a^m \times a^m \times a^m$ . . . . . to n factors is a^{mn}, whether m be whole or fractional. In the same manner the nth power of a^3b^2c is $a^{3n}b^{2n}c^n$, &c.

If the quantity have a coefficient, that coefficient must be raised to the proposed power, and prefixed to the power of the letters.

NOTE. If the quantity to be involved be negative, the signs of the *even* powers must evidently be positive, and those of the *odd* powers negative.

EXAMPLES.

The square of $2ax$ is $4a^2x^2$.

The fourth power of $6a^2x$ is $1296a^8x^4$.

The third power of $-a^{\frac{1}{2}}b^{\frac{1}{3}}c$ is $-a^{\frac{3}{2}}bc^3$.*

The fourth power of $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ is x^2y^{-2}.

The sixth power of $2\frac{a^2}{b}$ is $64\frac{a^{12}}{b^6}$.

The nth power of $3a^2x^3$ is $3^na^{2n}x^{3n}$.

The fifth power of $a\sqrt[5]{xy}$ is a^5xy.

The fifth power of $\frac{x^{\frac{1}{2}}}{y^3}$ is

The seventh power of $-a^{-2}x^{-\frac{1}{2}}$ is

The fourth power of $-\frac{a^m}{2x^n}$ is

The nth power of a^mx^m is

(37.) When the quantity is compound, the involution is performed by actual multiplication.

---

* The quantity $a^{\frac{3}{2}}$ signifies the third power of $a^{\frac{1}{2}}$. The denominator of every such fractional exponent always expresses, agreeably to the notation explained in the definitions, the *root*, and the numerator the *power* of that root. If, for instance, a represented 4, then $a^{\frac{1}{2}}$ would represent its second or square root, viz. 2; and $a^{\frac{3}{2}}$ would express the cube or third power of this root, and would therefore signify 8.

EXAMPLES.

What is the fourth power of $a+b$?

$$
\begin{array}{rl}
 & a + b \\
 & a + b \\ \hline
 & a^2 + ab \\
 & \quad\quad ab + b^2 \\ \hline
\text{The square} & a^2 + 2ab + b^2 \\
 & a \;+ b \\ \hline
 & a^3 + 2a^2b + ab^2 \\
 & \quad\quad a^2b + 2ab^2 + b^3 \\ \hline
\text{The cube} & a^3 + 3a^2b + 3ab^2 + b^3 \\
 & a \;+ b \\ \hline
 & a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
 & \quad\quad a^3b + 3a^2b^2 + 3ab^3 + b^4 \\ \hline
\text{The fourth power} & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ \hline
\end{array}
$$

What is the square of $a + b + c$?

$$
\begin{array}{l}
a + b + c \\
a + b + c \\ \hline
a^2 + ab + ac \\
\quad\quad ab + b^2 + bc \\
\quad\quad\quad\quad ac + bc + c^2 \\ \hline
\end{array}
$$

$$a^2 + 2ab + 2ac + b^2 + 2bc + c^2 = a^2 + 2ab + b^2 + 2c\,(a + b) + c^2 = (a + b)^2 + 2c\,(a + b) + c^2.$$

In the involution of $a+b$ we observed that its square was equal to the square of a + the square of b + twice the product of a, b; and, in the square of $a + b + c$, by considering $a + b$ as one term, we have the same property, viz. it is equal to the square of $(a+b)$ + the square of c + twice the product of $(a + b)\,c$, as we have just seen; and it might also be shown, in a similar way, that the square

of a quantity of four terms has the same property, by separating the three first terms, and considering them as a single term; and so on of any polynomial whatever.

Required the cube of $(a-x)$.

Ans. $a^3-3a^2x+3ax^2-x^3$.

Required the square of $4ax+x+1$.

Ans. $16a^2x^2+8ax^2+8ax+x^2+2x+1$.

Required the 4th power of $(a-x)$.

Ans. $a^4-4a^3x^3+6a^2x^2-4ax^3+x^4$.

Required the 4th power of $\sqrt{x^2+y^2}$.

Ans. $x^4+2x^2y^2+y^4$.

Required the 6th power of $\sqrt{a+x}$.

Ans. $a^3+3a^2x+3ax^2+x^3$.

Required the 5th power of $\sqrt[4]{x-y}$.

Ans. $(x-y)\sqrt[4]{x-y}$.

EVOLUTION.

(38.) EVOLUTION is the extracting of roots.

The evolution of simple quantities is represented by indices, similarly to involution, and if the simple quantity have already an index, or if it be composed of factors having indices, the operation of evolution is performed by *dividing* the index or indices, being the reverse of the operation of involution: thus, the nth root of a^{mn} is a^m, the nth root of a^r is $a^{\frac{r}{n}}$, the nth of a^rb^s is $a^{\frac{r}{n}}b^{\frac{s}{n}}$, &c. Hence the roots of quantities are properly expressed by means of *fractional* indices; for the cube root of a^2 is $a^{\frac{2}{3}}$, and the cube root of a or a^1 is $a^{\frac{1}{3}}$; also the fourth root of ab^3 is $a^{\frac{1}{4}}b^{\frac{3}{4}}$, &c.

It likewise appears that, since the division of the powers of the

same quantity is performed by subtracting their indices, when the divisor is greater than the dividend, the quotient must be a quantity with a *negative* index: thus,

$$\frac{a^2}{a^3} = a^{2-3} = a^{-1};\ \frac{a^2}{a^4} = a^{2-4} = a^{-2}, \text{ \&c.}$$

$$\text{also, } \frac{a^n}{a^n} = a^{n-n} = a^0, \text{ but } \frac{a^n}{a^n} = 1,$$

whence results this singular property, that a^0 is always $= 1$, whatever be the value of a. It also follows that

$$\frac{a^0}{a} \text{ or } \frac{1}{a} = a^{0-1} = a^{-1};\ \frac{a^0}{a^2} = \frac{1}{a^2} = a^{-2}, \text{ \&c.}$$

$$\text{or } \frac{a^0}{a^n} = \frac{1}{a^n} = a^{-n}.$$

Hence, *generally*, both powers and roots, as also the reciprocals* of both powers and roots are correctly represented by means of exponents or indices.

NOTE. Since the even powers of all quantities, whether positive or negative, are alike positive, (Art. 36, NOTE,) it follows that the even roots of all positive quantities may be either positive or negative; but the odd roots of a negative quantity must be negative, and, of a positive quantity, positive.

EXAMPLES.

The cube root of a^2x^6 is $a^{\frac{2}{3}}\,x^2$.

The 5th root of $\dfrac{1}{a^2b^3}$ is $\dfrac{1}{a^{\frac{2}{5}}b^{\frac{3}{5}}}$, or $a^{-\frac{2}{5}}b^{-\frac{3}{5}}$.

The square root of $\dfrac{a^2x^3}{b^3c^4d^5}$ is $\dfrac{ax^{\frac{3}{2}}}{b^{\frac{3}{2}}c^2d^{\frac{5}{2}}}$, or $ax^{\frac{3}{2}}b^{-\frac{3}{2}}c^{-2}d^{-\frac{5}{2}}$.

The cube root of $-8a^{-3}b^6x^{-2}$ is $-2a^{-1}b^2x^{-\frac{2}{3}}$ or $-\dfrac{2b^2}{ax^{\frac{2}{3}}}$

---

* The reciprocal of any quantity is unity divided by that quantity; thus, $\dfrac{1}{a^2}$ is the reciprocal of a^2, $\dfrac{1}{a+x}$ is the reciprocal of $a+x$, &c.

The 4th root of $\frac{16a^2b}{81c^2d^3}$ is $\frac{2a^{\frac{1}{2}}b^{\frac{1}{4}}}{3c^{\frac{1}{2}}d^{\frac{3}{4}}}$, or $2a^{\frac{1}{2}}b^{\frac{1}{4}} \cdot 3^{-1}c^{-\frac{1}{2}}d^{-\frac{3}{4}}$.

The square root of $\frac{4}{a^{\frac{1}{2}}b^3}$ is

The 4th root of $a^{-8}b^{-\frac{1}{2}}c$ is

The cube root of $-27a^2b^{\frac{1}{3}}x^{-3}$ is

The 5th root of $\frac{ab^{10}c^5}{d^2e^{\frac{1}{3}}m^2}$ is

The cube root of $\frac{a^{-1}}{b^n x^{\frac{1}{n}}}$ is

To Extract the Square Root of a Compound Quantity.

(39.) Arrange the terms according to the dimensions of some letter, and extract the root of the first term, which must always be a square; place this root in the quotient, subtract its square from the first term, and there will be no remainder.

Bring down the two next terms for a dividend, and put twice the root just found in the divisor's place, and see how often this is contained in the first term of the dividend, and connect the quotient both to the last found root and to the divisor, which will now be completed. Multiply the complete divisor by the term last placed in the quotient, subtract the product from the dividend, and to the remainder connect the two next terms in the compound quantity, and proceed as before; and so on till all the terms are brought down.

The reason of the above method of proceeding will appear obvious from considering that, as the square of $a+b$ is $a^2+2ab+b^2$ (Art. 37), the square root of $a^2+2ab+b^2$ must be $a+b$. Now a is the root of the first term, whose square being subtracted, leaves $2ab+b^2$, the first term of which divided by $2a$ gives b, the other part of the root, which, connected to $2a$, completes the divisor $2a+b$, and this multiplied by b, the term last found, gives $2ab+b^2$, which finishes the operation; and these several steps agree with the rule.

$$\begin{array}{rl}
 & a^2+2ab+b^2(a+b \\
 & a^2 \\
 & \overline{} \\
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
 & \overline{\;*\quad\;\;*\;} \\
\end{array}$$

If the root consist of three terms, $a+b+c$, its square will be $(a+b)^2+2c(a+b)+c^2$ (Art. 37), and we may return from this square to its root in a similar manner, viz. by finding first a, and then b, as above, and then deriving c from $(a+b)$ in the same way that b was derived from a; which is also according to the rule, and the same might be shown when the root consists of four, or a greater number of terms.

$$\begin{array}{l|l}
 & a^2+2ab+b^2+2c(a+b)+c^2\,(a+b+c \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
2(a+b)+c & 2c(a+b)+c^2 \\
 & 2c(a+b)+c^2 \\
\hline
 & * \quad * \quad *
\end{array}$$

EXAMPLES.

1.

$$\begin{array}{l|l}
 & 9x^4-12x^3+16x^2-8x+4\,(3x^2-2x+2 \\
 & 9x^4 \\
\hline
6x^2-2x & -12x^3+16x^2 \\
 & -12x^3+4x^2 \\
\hline
6x^2-4x+2 & 12x^2-8x+4 \\
 & 12x^2-8x+4 \\
\hline
 & * \quad * \quad *
\end{array}$$

2.

$$\begin{array}{l|l}
 & 4x^6+12x^5+5x^4-2x^3+7x^2-2x+1\,(2x^3+3x^2-x+1 \\
 & 4x^6 \\
\hline
4x^3+3x^2 & 12x^5+5x^4 \\
 & 12x^5+9x^4 \\
\hline
4x^3+6x^2-x & -4x^4-2x^3+7x^2 \\
 & -4x^4-6x^3+x^2 \\
\hline
4x^3+6x^2-2x+1 & 4x^3+6x^2-2x+1 \\
 & 4x^3+6x^2-2x+1 \\
\hline
 & * \quad * \quad * \quad *
\end{array}$$

3. Extract the square root of $4x^4-16x^3+24x^2-16x+4$.

Ans. $2x^2-4x+2$.

4. Extract the square root of $16x^4+24x^3+89x^2+60x+100$.

Ans. $4x^2+3x+10$.

5. Extract the square root of $1+x$.

Ans. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}$, &c.

6. Extract the square root of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$.

Ans. $3x^3-2x^2+x-4$.

(40.) From the above method of extracting the square root of algebraical quantities, is derived that for the extraction of the square root of numbers, usually given in books on arithmetic. In order to extract the square root of a number, it is necessary, first, to ascertain how many figures the root sought will consist of; which may be done by placing a dot over every alternate figure, commencing with the units: the number of these dots will be the number of figures in the root; for since the square root of 100 is 10, the square root of every whole number less than 100 must consist of but one figure; and since the square root of 100 is 10000, it follows that the square root of every number between 100 and 10000 must consist of but two figures; likewise, since the square root of 1000000 is 1000, the square root of every number between 10000 and 1000000 must consist of but three figures, &c.; and in each of these cases the number of dots will be 1, 2, 3, &c. &c. respectively.

Let it be required to find the square root of 56644.

Here the points indicate that the root consists of three figures, and the greatest square contained in 5 is 4, therefore the first figure of the root is 2, whose square taken from 5 leaves 1, to which the two next figures 66 being connected, gives 166 for the first dividend. Put 4, the double of the root figure, in the divisor's place, and it is contained in 16, 4 times; but, upon trial, 4 is found rather too great, the next figure of the root is therefore 3, which number being placed in the divisor com-

$$
\begin{array}{r|l}
 & \dot{5}6\dot{6}4\dot{4}\,(238 \\
 & 4 \\
\hline
43 & 166 \\
 & 129 \\
\hline
468 & 3744 \\
 & 3744 \\
\hline
\end{array}
$$

pletes it; and the product of this divisor and last root figure taken from the dividend, leaves 37, to which the remaining two figures are connected, and the same operation repeated. To exhibit, however, more clearly the similarity between this and the algebraical process, let the figures of the number 56644 be represented according to their values in the arithmetical scale; thus, the value of the first figure 5 is 50000, that of the second 6000, of the third 600, of the fourth 40, and of the last 4. Now, as it is necessary that the first term should be a square, and as in this case it is not, it will be proper to substitute for 50000, 40000+10000, 40000 being the greatest square contained in it; the operation will then be as follows:

$$\begin{array}{r|l}
 & 40000 + 10000 + 6644\ (200 + 30 + 8 \\
 & 40000 \qquad\qquad\qquad\qquad\quad \text{or} \\
 & \overline{\qquad\qquad} \qquad\qquad\qquad\qquad 238 \\
400 + 30 & 10000 + 6000 \\
\text{or} & \quad \text{or} \\
430 & \quad 16000 \\
 & \quad 12900 \\
 & \quad \overline{\qquad\qquad} \\
400 + 60 + 8 & 3100 + 600 + 40 + 4 \\
\text{or} & \qquad \text{or} \\
468 & \qquad 3744 \\
 & \qquad 3744 \\
 & \qquad \overline{\qquad\quad}
\end{array}$$

To Extract the Cube Root of a Compound Quantity.

(41.) Arrange the terms according to the dimensions of some letter, and extract the root of the first term, which must be a cube; place this root in the quotient, subtract its cube from the first term, and there will be no remainder.

Bring down the three next terms for a dividend, and put three times the square of the root just found in the divisor's place, and see how often it is contained in the first term of the dividend, and the quotient is the next term of the root. Add three times the product of the two terms of the root, plus the square of the last term, to

the term already in the divisor's place, and the divisor will be completed.

Multiply the complete divisor by the last term of the root, subtract the product from the dividend, and to the remainder connect the three next terms, and proceed as before.

For (by Art. 37,) the cube of $a+b$ is

$$a^3 + 3a^2b + 3ab^2 + b^3;$$

and, from having the cube given, its root is found by the following process, being the same as that directed above, and which, after what has been said of the square root, does not seem to need any further explanation.

$$\begin{array}{r|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 \,(\, a + b \\
 & a^3 \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\
\hline
 & * \quad * \quad *
\end{array}$$

If the root consist of three terms, a, b, c, they may be obtained by first finding a and b, as above, and then deriving c from $(a+b)$ in the same manner that b was derived from a.

EXAMPLES.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{r|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \;(x^2-2x+1) \\
 & x^5 \\
\hline
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-\;\;8x^3 \\
\hline
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1 \\
\hline
 & * \quad * \quad * \quad *
\end{array}$$

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$
\begin{array}{r|l}
 & x^6+6x^5-40x^3+96x-64\ (x^2+2x-4 \\
 & x^6 \\ \hline
3x^4+6x^3+4x^2 & 6x^5-40x^3 \\
 & 6x^5+12x^4+8x^3 \\ \hline
3x^4+12x^3-24x+16 & -12x^4-48x^3+96x-64 \\
 & -12x^4-48x^3+96x-64^{*} \\ \hline
 & * \quad * \quad * \quad * \\ \hline
\end{array}
$$

3. Extract the cube root of $8x^3+36x^2+54x+27$.

Ans. $2x+3$.

4. Extract the cube root of $27b^6-54x^5+63x^4-44x^3+21x^2-6x+1$.

Ans. $3x^2-2x+1$.

5. Extract the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$.

Ans. $a+b+c$.

From the foregoing method of extracting the cube root algebraically, may be derived the numerical process for the cube root given in books of arithmetic. But this tedious operation is now entirely superseded by the easy and concise method which we have given in our chapter on Cubic Equations, contained in the *Chapter on the Theory of Equations*, which forms a supplement to the present volume.

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* In this example only two terms are brought down each time, instead of three, because in the proposed expression there are two terms absent, viz. those containing x^4 and x^2. If we write the expression thus,

$$x^6+6x^5\pm 0x^4-40x^3\pm 0x^2+96x-64,$$

then three terms will, in effect, have been brought down, as in the preceding example, since $0x^4$ and $0x^2$ are each $=0$.

CHAPTER II.

ON SIMPLE EQUATIONS.

(42.) An Equation is an algebraical expression of equality between two quantities.

Thus, $4+8=12$ is an equation, since it expresses the equality between $4+8$ and 12; also, if there be an equality between $a-b+c$ and $f+g-h$, then $a-b+c=f+g-h$ expresses that equality, and is therefore an equation.

(43.) A Simple Equation, or an equation of the first degree, is that which contains the unknown quantity *simply;* that is, without any of its powers except the first.

(44.) A Quadratic Equation, or an equation of the second degree, is that which contains the square, but no higher power, of the unknown quantity.

(45.) An equation of the third, fourth, &c. degree, is one in which the highest power of the unknown quantity is the third, fourth, &c. power.

(46.) And in general an equation, in which the mth is the highest power of the unknown quantity, is called an equation of the mth degree.

Note. Each of the two members of an equation is called a side.

AXIOMS.

(47.) 1. If equal quantities be either increased or diminished by the same quantity, the results will be equal; or, in other words, if each side of an equation be either increased or diminished by the same quantity, the result will be an equation.

2. If each side of an equation be either multiplied or divided by the same quantity, the result will be an equation.

3. If each side of an equation be either involved to the same power, or evolved to the same root, the result will be an equation.

4. And generally, whatever operations be performed on one side of an equation, if the same operations be performed on the other side, the result will be an equation.

PROPOSITION.

(48.) Any term on one side of an equation may be transposed to the other, provided its *sign* be changed.

For let $x+a-b=c+d$ be an equation, and add b to both sides: then (by axiom 1), $x+a-b+b=c+d+b$, that is $x+a=c+d+b$, where b is transposed from the left to the right hand side of the equation, and its sign changed. Again, subtract a from each side of this last equation, then $x+a-a=c+d+b-a$; that is, $x=c+d+b-a$, where a is transposed, and its sign changed; and in the same manner may any other term be transposed.

EXAMPLES.

Transpose all the terms containing the unknown quantity x, in the following equations, to the left hand side, and the known terms to the right.

1. Given $4x+12=2x-x+21$.

   Here $4x-2x+x=21-12$, the terms being transposed as required.

2. Given $\frac{x}{2}=10-\frac{x}{4}+\frac{x}{3}$.

   Here $\frac{x}{2}+\frac{x}{4}-\frac{x}{3}=10$.

3. Given $14-x=6x+22$.
4. Given $\frac{4+x}{3}-x=\frac{6(x-2)}{5}-8$.
5. Given $3x+7=23-5x+\frac{4x-1}{2}$.
6. Given $ab+\frac{ax}{b}=a(x-b)+b$.
7. Given $5x+8-\frac{1}{2}x=6-\frac{3}{2}x+ax$.
8. Given $(2+x)(a-3)=13-2ax$.
9. Given $(a+b)(c-x)=(x-a)b$.

PROBLEM I.

To clear an Equation of Fractions.

(49.) 1. Multiply each numerator by all the denominators, except its own, and the result will be an equation free from fractions: or,

2. Multiply every term by a common multiple of the denominators, and the denominators may then be expunged. If the *least* common multiple be used, the resulting equation will be in its lowest terms.*

The reason of the first of these methods is plain; for multiplying the numerator of a fraction by its denominator is the same, in effect, as expunging the denominator; and multiplying every numerator by all the denominators, except its own, which is left out, or expunged, is the same as multiplying every term by the product of the denominators; each side of the equation is, therefore, multiplied by the same quantity, and therefore they are equal (axiom 2).

The second method is equally obvious: for each term being multiplied by a multiple of the denominators, the numerator of each fraction becomes divisible by its own denominator, and therefore the denominators may be expunged.

EXAMPLES.

1. Clear the equation $\frac{1}{7}x + \frac{2}{3}x = 12 - \frac{3}{4}x$ of fractions.

By the first method, the equation cleared of fractions is

$$12x + 56x = 1008 - 63x.$$

2. Clear the equation $\frac{x+6}{2} - 26 = \frac{5x}{4} + 2.$

Here 4 is evidently the least common multiple of the denominators;

∴ multiplying by 4,

$$2x + 12 - 104 = 5x + 8.$$

3. Clear the equation $\frac{4(x+3)}{5} - \frac{21}{4} = \frac{x}{6} - \frac{6x-8}{7} + 2.$

4. Clear the equation $\frac{2x+1}{3} - \frac{1}{7} = \frac{3x+5}{x-1}.$

5. Clear the equation $\frac{ax+b}{c} - \frac{a}{b} = \frac{cx+d}{ex}.$

6. Clear the equation $\frac{ax}{a+x} + b - \frac{a+x}{x} = 0.$

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* If a common multiple be evident from inspection, this last method will generally be the best; but if not, the other method will be preferable.

7. Clear the equation $\frac{x+3}{4}+6=\frac{2x-1}{3}+\frac{1}{2}$.

8. Clear the equation

$$\frac{4a(x+1)}{3}+\frac{2a(x-2)}{a}=\frac{a+x}{2a}+\frac{3}{2}.$$

9. Clear the equation

$$a+\frac{3a}{a+x}+2=\frac{4ax}{a-x}+\frac{x}{a^2-x^2}.$$

10. Clear the equation

$$\frac{ax}{a-x}+\frac{x}{a+x}=\frac{a}{a-x}+\frac{1}{a^2-x^2}.$$

11. Clear from fractions the equation

$$\frac{3-x}{2}+\frac{3}{5}=\frac{1}{20}+\frac{x-8}{10}.$$

12. Clear from fractions the equation

$$\frac{a+x}{\sqrt{a^2-x^2}}+\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{\sqrt{a-x}}{\sqrt{a+x}}.$$

PROBLEM II.

To clear an Equation of Radical Signs.

(50.) Transpose all the terms, except that under the radical, to one side of the equation.

Raise each side to the power denoted by the radical, and it will disappear. If there be more than one radical, this operation must be repeated.

EXAMPLES.

1. It is required to free from radicals the equation

$$\sqrt{a+x}+a=b,$$

By transposition, $\sqrt{a+x}=b-a$;

and by squaring each side, $a+x=b^2-2ab+a^2$.

2. It is required to free from radicals the equation

$$\sqrt{3x+\sqrt{x-6}}-2=3x.$$

By transposition, $\sqrt{3x+\sqrt{x-6}}=3x+2$;

and by squaring each side,

$$3x + \sqrt{x-6} = 9x^2 + 12x + 4;$$

or by transposing,

$$\sqrt{x-6} = 9x^2 + 12x + 4 - 3x = 9x^2 + 9x + 4;$$

whence, by squaring again,

$$x - 6 = (9x^2 + 9x + 4)^2 = 81x^4 + 162x^3 + 153x^2 + 72x + 16.$$

3. It is required to free from radicals the equation

$$\sqrt[3]{a^2x + \sqrt{a^3x^3}} = b.$$

By cubing $a^2x + \sqrt{a^3x^3} = b^3$;

and by transposition, $\sqrt{a^3x^3} = b^3 - a^2x$;

and by squaring $a^3x^3 = (b^3 - a^2x)^2 = b^6 - 2a^2b^3x + a^4x^2$.

4. It is required to free from radicals the equation

$$\sqrt{x+7} = \sqrt{x} + 1.$$

By squaring, in order to clear the first side,

$$x + 7 = x + 2\sqrt{x} + 1,$$

and by transposing,

$$x + 7 - x - 1 = 2\sqrt{x};$$

that is,

$$6 = 2\sqrt{x} \therefore 3 = \sqrt{x},$$

and by squaring, we have finally

$$9 = x.$$

5. Clear from radicals the equation

$$\sqrt{3-x} + 6 = 8 + x.$$

6. Clear from radicals the equation

$$\sqrt{x} - 2 = 4 - 3\sqrt{x}.$$

7. It is required to free from radicals the equation

$$24 + \sqrt{ax+b} = 2x - a.$$

8. It is required to free from radicals the equation

$$a + \sqrt{-a + \sqrt{x+2}} = 3.$$

9. It is required to free from radicals the equation

$$\sqrt[3]{a + \sqrt{2ax}} = x.$$

10. It is required to free the equation

$$\sqrt{1 + \sqrt{x} + \sqrt{ax}} = 2 \text{ from radicals.}$$

11. It is required to free from radicals the equation

$$\sqrt{a-x}+2=6-\sqrt{x}.$$

12. It is required to free from radicals the equation

$$\sqrt[3]{x-4}-1=\sqrt[3]{2+\sqrt{x}}-1.$$

PROBLEM III.

To solve a Simple Equation containing but one Unknown Quantity.

(51.) Clear the equation of fractions and radicals, if there be any.

Bring the unknown terms to one side of the equation, and the known terms to the other.

Collect each side into one term, and the unknown quantity, with a known coefficient, will form one side of the equation, and a known quantity the other side.

Divide each side by the coefficient of the unknown quantity, and the value of the unknown will be exhibited.

NOTE. Before performing any of the above operations, the equation may sometimes be previously simplified by the application of the 1st or 2d axioms, as will be seen in some of the following solutions.

EXAMPLES.

1. Given $4x+26=59-7x$, to find the value of x.

By transposition, $4x+7x=59-26$;

collecting the terms $11x=33$.

$\therefore$ dividing by 11, and we get $x=\frac{33}{11}=3$.

2. Given $\dfrac{x}{3}+6x=\dfrac{4x-2}{5}$, to find the value of x.

Clearing the equation $5x+90x=12x-6$;

and by transposition, $5x+90x-12x=-6$.

or collecting the terms $83x=-6$;

$\therefore$ dividing by 83, $x=-\frac{6}{83}$.

3. Given $\dfrac{3x+4}{5}-\dfrac{7x-8}{2}=\dfrac{x-16}{4}$, to find the value of x.

Here we immediately perceive that 20 is the least common multiple of the denominators;

∴ multiplying every term by 20,

$$12x + 16 - 70x + 30 = 5x - 80;$$

and by transposition,

$$12x - 70x - 5x = -80 - 16 - 30;$$

or collecting the terms $-63x = -126$;

∴ dividing by -63, $x = \dfrac{-126}{-63} = 2$.

4. Given $\dfrac{x+3}{7} - \frac{4}{5} = \dfrac{2(x-1)}{3} - \frac{9}{5}$, to find the value of x.

By transposition, $\dfrac{x+3}{7} - \dfrac{2(x-1)}{3} = \frac{4}{5} - \frac{9}{5} = -1$;

and clearing the equation, $3x+9-14x+14 = -21$:

or by transposing, $3x - 14x = -21 - 9 - 14$;

and collecting the terms $-11x = -44$;

∴ dividing by -11, $x = \dfrac{-44}{-11} = 4$.

5. Given $\dfrac{6x-4}{3} - 2 = \dfrac{18-4x}{3} + x$, to find the value of x.

Multiplying every term by 3,

$$6x - 4 - 6 = 18 - 4x + 3x;$$

and by transposition, $6x + 4x - 3x = 18 + 4 + 6$:

or collecting the terms $7x = 28$;

∴ dividing by 7, $x = \frac{28}{7} = 4$.

6. Given $\dfrac{3x+1}{3x} - \dfrac{3(x-1)}{3x+2} = \dfrac{9}{11x}$, to find the value of x.

Clearing the first side,

$$9x^2+9x+2-9x^2+9x = \frac{81x^2+54x}{11x} = \frac{81x+54}{11};$$

Or, collecting the terms on the first side,

$$18x + 2 = \frac{81x+54}{11};$$

and multiplying by 11, $198x + 22 = 81x + 54$:
or by transposing, $198x - 81x = 54 - 22$;
that is, $117x = 32$; $\therefore x = \frac{32}{117}$.

7. Given $\frac{x-2}{\sqrt{x}} = \frac{2\sqrt{x}}{3}$, to find the value of x.

Clearing the equation $3x - 6 = 2x$;
or, by transposing, $3x - 2x = 6$; that is, $x = 6$.

8. Given $x + \sqrt{2ax + x^2} = a$, to find the value of x.

By transposition, $\sqrt{2ax + x^2} = a - x$;
and squaring each side, $2ax + x^2 = a^2 - 2ax + x^2$:
or by transposing, $2ax + 2ax = a^2 + x^2 - x^2$;
that is, $4ax = a^2$, and $\therefore x = \frac{a^2}{4a} = \frac{a}{4}$.

9. Given $2\sqrt{a^2 + x^2} = 4(a - \frac{1}{2}x)$, to find the value of x.

By squaring each side, $4a^2 + 4x^2 = 16a^2 - 16ax + 4x^2$,
and subtracting $4x^2$, $4a^2 = 16a^2 - 16ax$ (axiom 1);
or dividing by $4a$, $a = 4a - 4x$ (axiom 2);
and by transposition, $4x = 4a - a$;

$$\text{and } \therefore x = \frac{3a}{4}.$$

10. Given $a + x = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$, to find the value of x.

By squaring each side,

$$a^2 + 2ax + x^2 = a^2 + x\sqrt{b^2 + x^2};$$

and subtracting a^2, $2ax + x^2 = x\sqrt{b^2 + x^2}$ (axiom 1);
then dividing by x, $2a + x = \sqrt{b^2 + x^2}$ (axiom 2);
and squaring both sides, $4a^2 + 4ax + x^2 = b^2 + x^2$;
or subtracting x^2, $4a^2 + 4ax = b^2$ (axiom 1);
and by transposition, $4ax = b^2 - 4a^2$;

$$\therefore x = \frac{b^2 - 4a^2}{4a}.$$

11. Given $\frac{4(x+2)}{3} - 1 = \frac{3x+1}{2}$, to find the value of x.

Ans. $x = 7$.

2. Given $\frac{x-1}{7} + \frac{x+4}{3} = x - 3$, to find the value of x.

Ans. $x = 8$.

13. Given $\frac{x}{2} - \frac{x}{3} + 5 = \frac{6(x+2)}{8}$, to find the value of x.

Ans. $x = 6$.

14. Given $\frac{2}{x+2} + \frac{x}{4} = \frac{x^2+1}{4x}$, to find the value of x.

Ans. $x = \frac{2}{7}$.

15. Given $\frac{1}{a^2 - x^2} - a = \frac{ax}{a-x} + \frac{a}{a+x}$, to find the value of x.

Ans. $x = \frac{a^2+a^3-1}{a-a^2}$.

16. Given $\frac{(a-b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$, to find the value of x.

Ans. $x = \frac{3a(b+4)}{6(a-b)+4}$.

17. Given $\frac{2}{3}x^2 + \frac{1}{2}x = x + \frac{x^2+x}{4}$, to find the value of x.

Ans. $x = 1\frac{4}{5}$.

18. Given $4abx^2 = \frac{3ax^2 - 2bx + ax}{3}$, to find the value of x.

Ans. $x = \frac{a-2b}{12ab-3a}$.

19. Given $21 + \frac{3x-11}{16} = \frac{5(x-1)}{8} + \frac{97-7x}{2}$, to find the value of x.

Ans. $x = 9$.

20. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 77$, to find the value of x.

Ans. $x = 60$.

21. Given $x + \frac{a}{b}x + \frac{c}{b}x = m$, to find the value of x.

Ans. $x = \frac{bm}{a+b+c}$.

22. Given $\sqrt{3x-1} = 2$, to find the value of x. Ans. $x = \frac{5}{3}$.

23. Given $\sqrt{x+x^2} = x + \frac{1}{3}$, to find the value of x.
Ans. $x = \frac{1}{3}$.

24. Given $3\sqrt{2x+6} + 3 = 15$, to find the value of x.
Ans. $x = 5$.

25. Given $\sqrt[3]{3x+13} - 4 = 0$, to find the value of x.
Ans. $x = 17$.

26. Given $\sqrt{x} + 3 = \sqrt{21+x}$, to find the value of x.
Ans. $x = 4$.

27. Given $\dfrac{\sqrt{a^2-y^2}}{\sqrt{a-y}} + y = a + 2y$, to find the value of y.
Ans. $y = 1 - a$.

28. Given $x + \sqrt{a-x} = \dfrac{a}{\sqrt{a-x}}$, to find the value of x.
Ans. $x = a - 1$.

29. Given $\sqrt{4+\sqrt{x^4-x^2}} = x - 2$, to find the value of x.
Ans. $x = 2\frac{1}{8}$.

30. Given $(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}} = 4\,(2+x)^{-\frac{1}{2}}$ to find the value of x.
Ans. $x = \frac{2}{3}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

(52.) In order to resolve a question algebraically, the first thing to be done is to consider attentively its conditions; then, having represented the quantity or quantities sought, by x, or x, y, &c. if we perform with it, or them, and the known quantities, the same operations that are described in the question, we shall finally obtain an equation from which the values of the assumed letters x, y, &c. may be determined. Instead of representing the unknown quantity by x or y, &c. it will sometimes be found more convenient to represent it by $2x$ or $2y$, or by $3x$, $3y$, &c. for the purpose of avoiding the introduction of fractional expressions in those cases where a half, a third, &c. of the unknown quantity is directed to be taken:

(see Question VII. following.) When we see by the conditions of the question that several different fractional parts of the unknown quantity will occur in the algebraical statement of those conditions, it will be advisable to represent the unknown by such a multiple of x or of y as will be actually divisible into the proposed parts. (See Question II. following.)

QUESTION I.

It is required to find a number, such, that if it be multiplied by 4, and the product increased by 3, the result shall be the same as if it were increased by 4, and the sum multiplied by 3.

Let x represent the number sought; then, if it be multiplied by 4, and the product increased by 3, there will result $4x+3$; but this result, according to the question, must be the same as $x+4$ multiplied by 3; hence we have this equation, viz.

$$4x+3=3x+12;$$

and by transposition, $4x-3x=12-3$;

that is, $x=9$, the number required.

QUESTION II.

It is required to find a number, such, that its third part increased by its fourth part, shall be equal to the number itself diminished by 10.

Let x represent the number.

Then, by the question, $\frac{x}{3}+\frac{x}{4}=x-10$;

or, clearing the equation, $4x+3x=12x-120$;

and by transposition, $4x+3x-12x=-120$;

that is, $-5x=-120$;

$$\therefore x=\frac{-120}{-5}=24, \text{ the number required.}$$

We might have avoided fractions in the statement of the conditions of this question, by representing the number sought not by x, but agreeably to the directions above, by such a multiple of x as would

really divide by 3 and 4. Choosing the least multiple, the process will be as follows;

Let $12x$ be the number.

Then, by the question, $4x+3x=12x-10$;

and by transposition, $4x+3x-12x=-10$;

that is, $-5x=-10 \therefore x=\frac{-10}{-5}=2$.

$\therefore 12x=24$, the number required.

QUESTION III.

A person left 350*l*. to be divided among his three servants, in such a way that the first was to receive double of what the second received, and the second double of what the third received. What was each person's share?

Let the share of the third be represented by x;

then that of the second was . . . . . $2x$

and that of the first . . . . $4x$

and, since the sum of their shares amounts to 350*l*.,

we have $x+2x+4x=350$;

or $7x=350$;

and $\therefore x=\frac{350}{7}=50$;

whence the share of the first was . . £50

of the second . . 100

of the third . . 200

QUESTION IV.

It is required to divide 160*l*. among three persons, in such a manner, that the first may receive 10*l*. more than the second, and the second 12*l*. more than the third.

Let the share of the third be x

then that of the second is $x+12$

and that of the first . . $x+12+10$;

and by the question, $x+x+12+x+12+10=160l.$;

that is, by addition and transposition, $3x=126$;

$$\text{whence, } x = \tfrac{126}{3} = 42:$$

$\therefore$ the share of the third is . £42
second . . 54
first . . . 64

QUESTION V.

A merchant has spirits at 9 shillings, and at 13 shillings, per gallon, and he wishes to make a mixture of 100 gallons that shall be worth 12 shillings per gallon. How many gallons of each must he take?

Suppose x to be the number of gallons at 9*s.*;

then $100 - x$ must be the number at 13*s.*;

also the value of the x gallons, at 9*s.*, is $9x$ shillings;

and of the $100 - x$, at 13*s.*, is $1300 - 13x$ shillings;

and the value of the whole mixture, at 12*s.*, is 1200*s.*:

$$\therefore 9x + 1300 - 13x = 1200;$$

$$\text{that is, } -4x = 1200 - 1300 = -100;$$

$$\text{consequently, } x = \frac{-100}{-4} = 25;$$

$\therefore$ there must be 25 gallons at 9*s.*
and $100 - 25 = 75$ . . 13*s.*

QUESTION VI.

How many gallons of spirits, at 9*s.* a gallon, must be mixed with 20 gallons at 13*s.*, in order that the mixture may be worth 10*s.* a gallon?

Let x be the number of gallons at 9*s.*, the value of which will be $9x$ shillings; also $x + 20$ will be the whole number of gallons in the mixture, the value of which, at 10*s.*, is $10x + 200$ shillings, now the value of the 20 gallons at 13*s.* is 260 shillings:

$$\therefore 9x + 260 = 10x + 200;$$

$$\text{and, by transposition, } 260 - 200 = 10x - 9x;$$

$$\text{that is, } 60 = x:$$

$\therefore$ there must be 60 gallons at 9*s.*, in order that the mixture, which will contain 80 gallons, may be worth 10*s.* a gallon.

QUESTION VII.

A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

Let $2x$ be the number of lbs. the body weighed;

then $9 + x =$ weight of the head;

and, since the body weighed as much as both head and tail, we have $2x = 9 + 9 + x$;

or, by transposition, $2x - x = 9 + 9$;

that is, $x = 18$;

$\therefore$ 36 lbs. = weight of the body,

$9 + x =$ 27 lbs. = ... head,

9 lbs. = ... tail,

The sum = 72 lbs. the whole weight of the fish.

QUESTION VIII.

If A can perform a piece of work in 12 days, and B can perform the same in 15 days, in what time will they finish it if they both work at it together?

Let x denote the number of days;

then $\frac{x}{12}$ is the part A can do in x days;

and $\frac{x}{15}$ is the part B can do in x days;

$$\therefore \frac{x}{12} + \frac{x}{15} = \text{the whole work } (= 1):$$

and, clearing the equation, $15x + 12x = 180$;

that is, $27x = 180$;

$\therefore x = \frac{180}{27} = 6\frac{2}{3}$;

$\therefore$ they will finish it in $6\frac{2}{3}$ days.

9. A person wishes to divide a straight line into 3 parts, so that the first part may be 3 feet less than the second, and the second 5 feet more than the third. Required the length of each part, that of the whole line being 37 feet?

Ans. the three parts are 12, 15, and 10 feet.

10. What number is that whose fifth part exceeds its sixth by 7?
Ans. 210.

11. Two persons, at the distance of 150 miles, set out to meet each other: one goes 3 miles, while the other goes 7. What part of the distance will each have travelled when they meet?
Ans. One 45 miles, and the other 105.

12. Divide the number 60 into two such parts, that their product may be equal to three times the square of the less?
Ans. The parts are 15 and 45.

13. Divide the number 45 into two parts, such that their product may be equal to the greater minus the square of the less.
Ans. The parts are $\frac{45}{46}$ and $\frac{2025}{46}$.

14. It is required to divide the number 36 into three such parts, that one half the first, one third the second, and one fourth the third, may be equal to each other.
Ans. The parts are 8, 12, and 16.

15. A person bought three parcels of books, each containing the same number, for 12*l.* 5*s.*; for the first parcel he gave at the rate of 5*s.* a volume, for the second 9*s.*, and for the third 10*s.* 6*d.* a volume. How many were there in each parcel? Ans. 10.

16. Find a number such that $\frac{1}{3}$ thereof increased by $\frac{1}{4}$ of the same, shall be equal to $\frac{1}{6}$ of it increased by 35. Ans. 84.

17. A post is $\frac{1}{3}$ in mud, $\frac{1}{4}$ in water, and ten feet above the water. Required the length of the post? Ans. 24 feet.

18. There is a cistern which can be supplied with water from three different cocks; from the first it can be filled in 8 hours, from the second in 10 hours, and from the third in 14 hours. In what time will it be filled if the three cocks be all set running together?
Ans. 3 hours 22 min. $24\frac{48}{83}$ sec.

19. A gentleman spends $\frac{2}{3}$ of his yearly income in board and lodging, $\frac{2}{3}$ of the remainder in clothes, and lays by 20*l.* a year. What is his income? Ans. 180*l.*

20. Two travellers set out at the same time, the one from London, in order to travel to York, and the other from York to travel to London; the one goes 14 miles a day, and the other 16. In what time

will they meet, the distance between London and York being 197 miles?

Ans. in 6 days, $13\frac{3}{5}$ hours

21. A person wishes to give 3*d.* a piece to some beggars, but finds he has not money enough by 8*d.*; but if he gives them 2*d.* a piece, he will have 3*d.* remaining. Required the number of beggars.

Ans. 11.

22. A gamester at play staked $\frac{1}{5}$ of his money, which he lost, but afterwards won 4*s.*; he then lost $\frac{1}{4}$ of what he had, and afterwards won 3*s.*; after this he lost $\frac{1}{6}$ of what he then had, and finding that he had but 1*l.* remaining, he left off playing. It is required to find how much he had at first? Ans. 1*l.* 10*s.*

23. A person mixed 20 gallons of spirits at 9*s.* a gallon, with 36 gallons at 11*s.* a gallon, and he now wishes to add such a quantity at 14*s.* a gallon as will make the whole worth 12*s.* a gallon. How much of this last must he add? Ans. 48 gallons.

24. If A can perform a piece of work in a days, and B can do the same in b days, in how many days will they have finished the work if they both work at it together?

Ans. in $\dfrac{ab}{a+b}$ days.

25. If A can perform a piece of work in a days, B in b days, C in c days, and D in d days, in how many days will they have finished the work, if they all work at it together?

Ans. in $\dfrac{abcd}{abc + abd + bdc + adc}$ days.

26. It is required to find a number such, that if it be increased by 7, the square root of the sum shall be equal to the square root of the number itself and 1 more. Ans. 9.

27. It is required to find two numbers, whose difference is 6, such, that if $\frac{1}{3}$ the less be added to $\frac{1}{5}$ the greater, the sum shall be equal to $\frac{1}{3}$ the greater diminished by $\frac{1}{5}$ the less. Ans. 2 and 8.

28. A labourer engages to work at the rate of 3*s.* 6*d.* a day, but on every day that he is idle he spends 9*d.*, and at the end of 24 days finds, that upon deducting his expenses, he has to receive 3*l.* 2*s.* 9*d.* How many days was he idle? Ans. 5 days.

29. A person being asked the hour, answered that it was between 5 and 6, and that the hour and minute hands were exactly together. What was the time? Ans. 27′ $16\frac{4}{11}$″ past 5.

30. A gentleman leaves 315*l.* to be divided among his 4 sons in the following manner, viz. the second is to receive as much as the first, and half as much more; the third is to receive as much as the first and second together, and $\frac{1}{3}$ as much more; and the eldest is to receive as much as the other three, and $\frac{1}{4}$ as much more. Required the share of each.

Ans. { The share of the 1st is 24*l.*, of the 2d 36*l.*, of the 3d 80*l.*, and of the 4th 175*l.*

PROBLEM IV.

To resolve Simple Equations containing two Unknown Quantities.

(53.) When there are given two independent simple equations, and two unknown quantities, the value of each unknown quantity may be obtained by either of the three following methods.

First Method.

(54.) Find the value of one of the unknown quantities in terms of the other and the known quantities, from the first equation, by the method already given. Find the value of the same unknown quantity from the second equation.

Put these two values equal to each other, and we shall then have a simple equation containing only one unknown quantity, which may be solved as before.

Thus, suppose $ax + by = c$; and $a'x + b'y = c'$.

Then, from the first equation, $x = \dfrac{c - by}{a}$;

and from the second . . $x = \dfrac{c' - b'y}{a'}$;

whence, equating these two values of x, $\dfrac{c - by}{a} = \dfrac{c' - b'y}{a'}$;

and clearing the equation, $a'c - a'by = ac' - ab'y$;

or, by transposition, $ab'y - a'by = ac' - a'c$;

$$\text{that is, } (ab' - a'b)\,y = ac' - a'c;$$

$$\therefore\ y = \frac{ac' - a'c}{ab' - a'b};$$

and this value being substituted in either of the above values of x gives

$$x = \frac{b'c - bc'}{ab' - a'b}.$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + 3y = 23 \\ 5x - 2y = 10 \end{array}\right\}$, to find the values of x and y.

From the first equation $x = \dfrac{23 - 3y}{2}$,

and from the second ... $x = \dfrac{10 + 2y}{5}$;

$$\therefore\ \frac{23 - 3y}{2} = \frac{10 + 2y}{5};$$

$$\text{or } 115 - 15y = 20 + 4y;$$

$$\text{and by transposition, } -15y - 4y = 20 - 115;$$

$$\text{or } -19y = -95;$$

$$\therefore\ y = \frac{-95}{-19} = 5;$$

$$\text{consequently, } x\left(= \frac{10 + 2y}{5}\right) = 4.$$

2. Given $\left\{\begin{array}{l} 5x + 2y = 45 \\ 4x + y = 33 \end{array}\right\}$, to find the values of x and y.

From the first equation $y = \dfrac{45 - 5x}{2}$,

and from the second ... $y = 33 - 4x$;

$$\therefore\ \frac{45 - 5x}{2} = 33 - 4x,$$

$$\text{or } 45 - 5x = 66 - 8x;$$

and by transposition, $8x - 5x = 66 - 45$;

that is, $3x = 21$,

$\therefore\ x = 7$, and $y\,(= 33 - 4x) = 5$.

3. Given $\left\{\begin{array}{l} 6x - 5y = 39 \\ 7x - 3y = 54 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 9$, and $y = 3$.

4. Given $\left\{\begin{array}{l} \frac{1}{4}x + \frac{1}{2}y = 7 \\ \frac{1}{3}x - \frac{1}{4}y = 2 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 12$, and $y = 8$.

5. Given $\left\{\begin{array}{r} 3x - \frac{1}{4}y = 7 \\ -\frac{1}{2}x + 2y = 14\frac{1}{2} \end{array}\right\}$, to find the values of x and y.

Ans. $x = 3$, $y = 8$.

Second Method.

(55.) Find the value of either of the unknown quantities from one of the equations, as in the preceding method. Substitute this value for its equal in the other equation, and we shall have an equation containing only one unknown quantity.

Thus, taking the same general example as before, viz. $ax + by = c$, and $a'x - b'y = c'$, if we substitute for x in the second equation, its value, $\frac{c - by}{a}$, as determined from the first, there will arise the equation

$$\frac{a'c - a'by}{a} + b'y = c'\text{; or } a'c - a'by + ab'y = ac'\text{;}$$

and by transposition, $ab'y - a'by = ac' - a'c$:

$$\left.\begin{array}{r} \therefore\ y = \dfrac{ac' - a'c}{ab' - a'b} \\ \text{and by substitution, } x = \dfrac{b'c - bc'}{ab' - a'b} \end{array}\right\}\text{ as before.}$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} 8x + 6y = 74 \\ 3x + 5y = 36 \end{array}\right\}$, to find the values of x and y.

From the first equation $8x = 74 - 6y$, or $x = \frac{74 - 6y}{8}$;

which value substituted in the second equation,

$$\text{gives } \frac{222 - 18y}{8} + 5y = 36;$$

$$\therefore\ 222 - 18y + 40y = 288:$$

$$\text{or } 40y - 18y = 288 - 222;$$

$$\text{that is, } 22y = 66;$$

$$\therefore\ y = \tfrac{66}{22} = 3, \text{ and } x \left(= \frac{74 - 6y}{8}\right) = 7.$$

2. Given $\left\{ \begin{array}{l} 7x+2y=30 \\ 5x+3y=34 \end{array} \right\}$, to find the values of x and y.

Ans. $x=2$, and $y=8$.

3. Given $\left\{ \begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=8 \\ \frac{1}{4}x-\frac{1}{3}y=1 \end{array} \right\}$, to find the values of x and y.

Ans. $x=12$, and $y=6$.

4. Given $\left\{ \begin{array}{l} \dfrac{2x+3y}{4}=5 \\ 2x=\dfrac{54-8y}{3} \end{array} \right\}$ to find the values of x and y.

Ans. $x=1$, and $y=6$.

Third Method.

(56.) Multiply or divide each of the given equations by such quantities, that the coefficient of one of the unknown quantities may be the same in both.

Destroy the identical terms by adding or subtracting these equations, and the result will be an equation containing only one unknown quantity.

NOTE. If multipliers, or divisors, do not readily present themselves, which will make the coefficient of any one of the unknowns the same in both equations, then each of the equations must be multiplied or divided by the coefficient of that unknown in the other equation, which we wish to exterminate.

Thus, taking our former general example, $ax + by = c$, and $a'x + b'y = c'$; if we multiply the second equation by a, and the first by a', in order that the coefficient of x may be the same in both equations, we shall have

$$aa'x + ab'y = ac'$$
$$aa'x + a'by = a'c$$

and subtracting $ab'y - a'by = ac' - a'c$

$$\therefore y = \frac{ac' - a'c}{ab' - a'b}$$

and by a similar process, $x = \dfrac{b'c - bc'}{ab' - a'b}$ as before.

EXAMPLES.

1\. Given $\left\{\begin{array}{l} 4x - 3y = 1 \\ 3x + 4y = 57 \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 3, and the second by 4, in order to equalize the coefficients of x, we have

$$12x - 9y = 3$$
$$12x + 16y = 228$$

and by subtracting $25y = 225$

$$\therefore y = \tfrac{225}{25} = 9;$$

$$\text{whence } x = \frac{3 + 9y}{12} = \frac{3 + 81}{12} = 7.$$

2\. Given $\left\{\begin{array}{l} 6x + 5y = 128 \\ 3x + 4y = 88 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 8$, and $y = 16$.

3\. Given $\left\{\begin{array}{r} 7x + 3y = 42 \\ -2x + 8y = 50 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 3$ and $y = 7$.

ADDITIONAL EXAMPLES.

1\. Given $\left\{\begin{array}{l} 5x + 7y = 201 \\ 8x - 3y = 137 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 22 \\ y = 13 \end{array}\right.$

2\. Given $\left\{\begin{array}{l} -3x + 8y = 29 \\ -4x + 6y = 20 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 1 \\ y = 4 \end{array}\right.$

3. Given $\left\{\begin{array}{r} 3x - \frac{1}{2}y = 3\frac{1}{2} \\ -x + 7y = 33 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2 \\ y = 5 \end{array}\right.$

4. Given $\left\{\begin{array}{r} \frac{1}{2}x + \frac{1}{3}y = 8 \\ \frac{1}{3}x - \frac{1}{5}y = -1 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 6 \\ y = 15 \end{array}\right.$

5. Given $\left\{\begin{array}{r} \frac{2x}{3} + 5y = 23 \\ 5x + \frac{7y}{4} = -6\frac{1}{4} \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = -3 \\ y = 5 \end{array}\right.$

6. Given $\left\{\begin{array}{l} \frac{x}{2} - 12 = \frac{y}{4} + 8 \\ \frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y-x}{4} + 27 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 60 \\ y = 40 \end{array}\right.$

7. Given $\left\{\begin{array}{l} x + y = a \\ x^2 - y^2 = b \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \frac{a^2 + b}{2a} \\ y = \frac{a^2 - b}{2a} \end{array}\right.$

8. Given $\left\{\begin{array}{l} b(x+y) = a(x-y) \\ x^2 - y^2 = c \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \frac{a+b}{2}\sqrt{\frac{c}{ab}} \\ y = \frac{a-b}{2}\sqrt{\frac{c}{ab}} \end{array}\right.$

(57.) QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

QUESTION I.

A vintner sold, at one time, 20 dozen of port wine, and 30 dozen of sherry, and for the whole received 120*l.*; and, at another time, he sold 30 dozen of port, and 25 of sherry, at the same prices as

before; and for the whole received 140*l.* What was the price of a dozen of each sort of wine?

Let x be the price of the port per dozen,

and y that of the sherry;

$$\left.\begin{array}{r}\text{then } 20x + 30y = 120 \\ \text{and } 30x + 25y = 140\end{array}\right\} \text{ or } \left\{\begin{array}{l}2x + 3y = 12 \\ 6x + 5y = 28\end{array}\right.$$

and multiplying the first equation by 3,

$$6x + 9y = 36$$

and subtracting $\underline{6x + 5y = 28}$

$$4y = 8$$

$$\therefore y = 2, \therefore 2l. \text{ is the price of the sherry;}$$

and $x\left(= \dfrac{12 - 3y}{2}\right) = 3$; $\therefore$ 3*l.* is the price of the port per dozen.

QUESTION II.

A farmer has 86 bushels of wheat at 4*s.* 6*d.* per bushel, with which he wishes to mix rye at 3*s.* 6*d.* per bushel, and barley at 3*s.* per bushel, so as to make 136 bushels, that shall be worth 4*s.* a bushel. What quantity of rye and of barley must he take?

Let x represent the number of bushels of rye,

and y the number of barley;

then $3\frac{1}{2}x$ shillings is the value of the rye,

$3y$ shillings .... barley,

and 387 shillings .... wheat,

now the value of the whole 136 bushels, at 4*s.*, is 544*s.*;

$$\therefore 3\tfrac{1}{2}x + 3y + 387 = 544;$$

$$\text{or } 3\tfrac{1}{2}x + 3y = 157;$$

also $x + y + 86 = 136$, $\therefore 3x + 3y = 150$ by transposing and multiplying by 3

and by subtraction, $\frac{1}{2}x = 7$

$$\therefore x = 14$$

and $y(= 136 - 86 - x) = 36$;

hence he must take 14 bushels of rye,

and 36 .... barley.

QUESTION III.

A person has 27*l.* 6*s.* in guineas and crown-pieces; out of which he pays a debt of 14*l.* 17*s.*, and finds he has exactly as many guineas left as he has paid away crowns; and as many crowns as he has paid away guineas. How many of each had he at first?

Suppose x the number of guineas paid away,

and y .... crowns ....;

then, by reducing to shillings, we have

$$\left.\begin{array}{rl} & 21x + 5y = 297 = \text{the amount paid away} \\ \text{and} & 5x + 21y = 249 = \text{the amount remaining} \end{array}\right\} \text{by the question;}$$

$\therefore$ multiplying the first equation by 5, and the second by 21,

$$\text{we have } \begin{cases} 105x + 25y = 1485 \\ 105x + 441y = 5229 \end{cases}$$

and by subtraction .... $416y = 3744$

$\therefore y = \frac{3744}{416} = 9 =$ no. of crowns paid away,

whence $x\left(= \dfrac{249-21y}{5}\right) = 12 =$ no. of guineas ....;

$\therefore$ he had at first 21 guineas and 21 crowns.

QUESTION IV.

There is a number, consisting of two digits, which is equal to four times the sum of those digits; and, if 9 be subtracted from twice the number, the digits will be inverted. What is the number?

Put $x =$ the first digit,

$y =$ the second;

$$\left.\begin{array}{r} \text{then the number is } 10x + y = 4x + 4y \\ \text{also } 20x + 2y - 9 = 10y + x \end{array}\right\} \text{by the question;}$$

from the first equation $6x = 3y$, or $y = 2x$,

and from the second $19x - 8y = 9$, or, substituting the above value of y in this equation, we have $19x - 16x = 9$, or $3x = 9$;

$\therefore x = 3$, and $y\,(= 2x) = 6$, $\therefore$ the number is 36.

5. A bill of 14*l.* 8*s.* was paid with half-guineas and crowns, and

twice the number of crowns was equal to three times the number of half-guineas. How many were there of each?

Ans. 16 half-guineas and 24 crowns.

6. There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 18 be added to it, the digits will be inverted. What is the number?

Ans. 24.

7. A man being asked the age of himself and son, replied, "If I were $\frac{1}{4}$ as old as I am + 3 times the age of my son, I should be 45; and if he were $\frac{1}{4}$ his present age + 3 times mine, he would be 111." Required their ages?

Ans. The father's age was 36, and the son's 12.

8. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? Ans. $\frac{4}{5}$.

9. A man and his wife could drink a barrel of beer in 15 days; but, after drinking together 6 days, the woman alone drank the remainder in 30 days. In what time could either alone drink the whole barrel?

Ans. The man could drink it in $21\frac{3}{7}$ days, and the woman in 50 days.

10. A farmer sold at one time 30 bushels of wheat and 40 bushels of barley, and for the whole received 13*l.* 10*s.*; and at another time he sold, at the same prices as before, 50 bushels of wheat and 30 bushels of barley, and for the whole received 17*l.* How much was each sort of grain sold at per bushel?

Ans. The wheat was sold at 5*s.*, and the barley at 3*s.* a bushel.

PROBLEM V.

To resolve Simple Equations containing three Unknown Quantities.

(58.) Either of the three methods given for the resolution of equations with two unknown quantities may be extended to this case; but, as the last of the three will generally be found preferable to the others, we shall therefore give it as our

1st Method.

(59.) Multiply, or divide, each of the two first equations by such quantities as will make the coefficients of one of the unknowns the same in both.

Destroy the identical terms, by adding or subtracting these equations, and the result will be an equation containing only two unknown quantities.

Perform a similar process on the first and third, or on the second and third, of the original equations, and there will result another equation containing only two unknown quantities; therefore we shall have two equations and two unknown quantities: hence this problem is reduced to the former.

After what has been done in Art. 12, there does not seem any necessity for showing the truth of this method in general terms; we shall therefore proceed to particular examples.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-\ \ z=63 \end{array}\right\}$ to find the values of x, y, and z.

Multiplying the first equa. by 2, $4x+8y-6z=44$
and subtracting the second, ... $4x-2y+5z=18$

there results..........(A).........$10y-11z=26$
Again, mult. the first equa. by 3, $6x+12y-9z=66$
and subtracting the third ...... $6x+7y-z=63$

there results .... $5y-8z=3$
and mult. this result by 2 .... $10y-16z=6$
which, subtracted from equation (A) $10y-11z=26$

gives .... $5z=20$

$\therefore z=4, y\left(=\dfrac{3+8z}{5}\right)=7$, and $x\left(=\dfrac{22-4y+3z}{2}\right)=3$.

2. Given $\left\{\begin{array}{l} 3x+2y-4z=8 \\ 5x-3y+3z=33 \\ 7x+\ \ y+5z=65 \end{array}\right\}$ to find the values of x, y, and z.

In this example it appears, from the coefficients, that y may be most readily exterminated;

$\therefore$ multiplying the first equation by 3, and the second by 2, they become

$$9x + 6y - 12z = 24$$
$$10x - 6y + \ 6z = 66$$

and by addition (A) .... $19x - \ 6z = 90$

Again, multiplying the third equation by 2, it becomes

$$14x + 2y + 10z = 130$$

and subtracting the first, $3x + 2y - \ 4z = \ 8$

there results .... $11x + 14z = 122$

and multiplying this last equation by 3, and equation (A) by 7, we have

$$33x + 42z = 366$$

and ...... $133x - 42z = 630$

and by addition .... $166x \qquad = 996$

$$\therefore x = 6$$

also $z\ (= \dfrac{90 - 19x}{-6}) = 4$, and $y\,(= 65 - 7x - 5z) = 3$.

3. Given $\left\{\begin{array}{l} 7x + 5y + 2z = 79 \\ 8x + 7y + 9z = 122 \\ x + 4y + 5z = 55 \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x = 4 \\ y = 9 \\ z = 3 \end{array}\right.$

4. Given $\left\{\begin{array}{l} 3x - 9y + 8z = 41 \\ -5x + 4y + 2z = -20 \\ 11x - 7y - 6z = 37 \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x = 2 \\ y = -3 \\ z = 1 \end{array}\right.$

5. Given $\left\{\begin{array}{l} x + \frac{1}{2}y + \frac{1}{3}z = 32 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12 \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x = 12 \\ y = 20 \\ z = 30 \end{array}\right.$

6. Given $\left\{\begin{array}{l}\frac{x+y}{3}+2z=\ \ 21\\ \frac{y+z}{2}-3x=-65\\ \frac{3x+y-z}{2}=\ \ 38\end{array}\right\}$ to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l}x=24\\ y=\ \ 9\\ z=\ \ 5\end{array}\right.$

Second Method.

(60.) Multiply the first equation by some undetermined quantity m, and the second by another, n;

Add the two equations, so multiplied, together, and from the sum subtract the third equation, and the result will be an equation containing all the three unknown quantities.

Then determine m and n, so that two of the unknowns may be destroyed, and the value of the other unknown will be obtained.

Thus, as a general example, let us take the three equations

$$\begin{aligned} ax + by\ \ + cz\ &= d,\\ a'x + b'y\ + c'z\ &= d',\\ a''x + b''y + c''z &= d''; \end{aligned}$$

then, multiplying the first by m, the second by n, adding the results, and subtracting the third equation, we have

$$(am + a'n - a'')x + (bm + b'n - b'')y + (cm + c'n - c'')z$$
$$= dm + d'n - d'';$$

now, in order to destroy x and y, put

$$\left.\begin{array}{r} am + a'n = a''\\ \text{and } bm + b'n = b'' \end{array}\right\}(A);$$

then, since the coefficients of x and y become 0, they vanish from the equation, which becomes simply

$$(cm+c'n-c'')z = dm+d'n-d'':$$

$$\therefore z = \frac{dm+d'n-d''}{cm+c'n-c''}$$

The values of m and n being found from equations (A) by last problem, are

$$m = \frac{a''b' - b''a'}{ab' - ba'};$$

$$\text{and } n = \frac{ab'' - ba''}{ab' - ba'};$$

and if these values be substituted in the above value of z, and the fractions, in both numerator and denominator, be brought to common denominators, we shall have

$$z = \frac{d(b'a'' - a'b'') + d(ab'' - ba'') - d''(ab' - ba')}{c(b'a'' - a'b'') + c(ab'' - ba'') - c''(ab' - ba')}.$$

In a similar manner may x and z be exterminated, and the value of y exhibited, by putting

$$am + a'n = a'',$$
$$cm + c'n = c'';$$

and y and z also may be exterminated, and the value of x exhibited, by putting

$$bm + b'n = b'',$$
$$cm + c'n = c'',$$

and proceeding as above. We shall therefore have

$$x = \frac{d(c'b'' - b'c'') + d'(bc'' - cb'') - d''(bc' - cb')}{a(c'b'' - b'c'') + a'(bc'' - cb'') - a''(bc' - cb')},$$

$$y = \frac{d(c'a'' - a'c'') + d'(ac'' - ca'') - d''(ac' - ca')}{b(c'a'' - a'c'') + b'(ac'' - ca'') - b''(ac' - ca')}.$$

$$z = \frac{d(b'a'' - a'b'') + d'(ab'' - ba'') - d''(ab' - ba')}{c(b'a'' - a'b'') + c'(ab'' - ba'') - c''(ab' - ba')}.$$

and, by substituting particular values in the above general expressions, any proposed example may be solved.

SCHOLIUM.

(61.) The method above given, has been introduced for the purpose of obtaining general values for the unknown quantities, that may apply to every particular example, by substituting in them the particular coefficients for the above general ones; this method being preferable for that purpose to the preceding one. When, however, the whole process is to be performed, particular examples are much more readily solved by the first method, and therefore we shall not

give any to this. It may here be further observed, that either of the two methods may be readily extended to equations containing four, or a greater number of unknown quantities, they being solved according to the first method, by equalizing the coefficients of the same unknown in any two equations, and then, by addition or subtraction, exterminating them one by one; or, according to the second method, by multiplying the first equation by m, the second by n, and the third by p, &c. to the last but one, subtracting the last equation from their sum, and then determining m, n, p, &c. so that all the unknowns in the resulting expression may vanish, except one, the value of which will become known.

Both these methods, however, from their giving only one unknown at a time, and their requiring a repetition of the process to determine each of the others, become at length very tedious; a circumstance which has induced several eminent mathematicians to attempt the discovery of a direct method, whereby the values of all the unknowns, in any number of equations of this kind, may be determined at once. The most successful of these has been Bezout, who, first in the Memoirs of the Academy of Sciences, and then in his *Theorie Générale des Equations Algébriques*, p. 172, gave a method, which is generally considered as the simplest that has yet appeared. It is as follows:

General Rule to calculate either all at once, or separately, the values of the unknown quantities in equations of the first degree, whether they be literal or numeral.

Let u, x, y, z, &c. be the unknowns, whose number is n, as also the number of the equations: Let a, b, c, d, &c. be the respective coefficients of the unknowns in the first equation; a', b', c', d', &c. the coefficients of those in the second; a'', b'', c'', d'', &c. of those in the third, &c.

Conceive the known term in each equation to be affected by some unknown quantity, represented by t; and form the product, $uxyzt$, of all the unknowns written in any order at pleasure; but this order, once determined, is to be preserved throughout the operation.

Change, successively, each unknown in this product for its coefficient in the first equation, observing to change the sign of each even term: the result is called the *first line*.

Change, in this first line, each unknown for its coefficient in the

second equation, observing, as before, to change the sign of each even term: the result is the *second line*.

Change, in this second line, each unknown for its coefficient in the third equation, still changing the sign of each even term, and the result is the *third line*.

Continue this process to the last equation, inclusively, and the last line that you obtain will give the values of the unknowns in the following manner:

Each unknown will have for its value a fraction, whose numerator will be the coefficient of the same unknown in the *last* or *n*th line; and the general denominator will be the coefficient of t, the unknown at first introduced.

Suppose we wish to find the values of x and y in the equations

$$ax+by+c=0, \text{ and } a'x+b'y+c'=0;$$

introducing t, these equations become

$$ax+by+ct=0, \text{ and } a'x+b'y+c't=0;$$

and forming the product, xyt, and then changing x into a, y into b, t into c, and changing the signs of the even terms, we have, for the *first line*, $ayt-bxt+cxy$; then changing x into a', y into b', t into c', and changing the signs, as before, we have for the *second line*

$$ab't-ac'y-a'bt+bc'x+a'cy-b'cx, \text{ or}$$
$$(ab'-a'b)t-(ac'-a'c)y+(bc'-b'c)x;$$

$$\text{whence } x=\frac{bc'-b'c}{ab'-a'b}, \text{ and } y=\frac{-(ac'-a'c)}{ab'-a'b}=\frac{a'c-ac'}{ab'-a'b};$$

and in the same manner may this method be applied to any number of equations whatever, containing an equal number of unknown quantities.*

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* Bezout does not give any demonstration of this rule in the work above referred to, and seems to have obtained it by induction. But a demonstration from *Laplace* may be seen in *Garnier's Analyse Algébrique*.

(62.) QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

QUESTION I.

If A and B can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days; in how many days can each alone perform the same work?

Let the number of days be x, y, and z, respectively:

$$\text{then } A \text{ can do } \frac{1}{x} \text{ of the whole in a day;}$$

$$B \ldots\ldots \frac{1}{y} \ldots\ldots\ldots\ldots\ldots\ldots;$$

$$C \ldots\ldots \frac{1}{z} \ldots\ldots\ldots\ldots\ldots\ldots;$$

and since A and B do the whole in 8 days,

$$\therefore \frac{8}{x}+\frac{8}{y}(=\text{the whole work})=1;$$

also, since A and C do the same in 9 days,

$$\therefore \frac{9}{x}+\frac{9}{z}(=\text{the whole work})=1;$$

$$\text{and in the same manner } \frac{10}{y}+\frac{10}{z}=1;$$

$\therefore$ dividing the first of these equations by 8, the second by 9, and the third by 10, we have

$$\frac{1}{x}+\frac{1}{y}=\tfrac{1}{8},$$

$$\frac{1}{x}+\frac{1}{z}=\tfrac{1}{9},$$

$$\frac{1}{y}+\frac{1}{z}=\tfrac{1}{10},$$

and subtracting the second equation from the first,

$$\frac{1}{y}-\frac{1}{z}=\tfrac{1}{8}-\tfrac{1}{9};$$

and adding the third to this, we get

$$\frac{2}{y} = \tfrac{1}{8} - \tfrac{1}{9} + \tfrac{1}{10} = \tfrac{41}{360}, \therefore y = \tfrac{720}{41} = 17\tfrac{23}{41};$$

also subtracting the third equation from the second, we have

$$\frac{1}{x} - \frac{1}{y} = \tfrac{1}{9} - \tfrac{1}{10}, \therefore \frac{1}{x} = \tfrac{1}{9} - \tfrac{1}{10} + \tfrac{41}{720} = \tfrac{49}{720};$$

whence $x = \tfrac{720}{49} = 14\tfrac{34}{49}$, and $\dfrac{1}{z} = \tfrac{1}{9} - \tfrac{49}{720} = \tfrac{31}{720}$,

$$\therefore z = \tfrac{720}{31} = 23\tfrac{7}{31}:$$

hence A can do the work in $14\tfrac{34}{49}$ days, B in $17\tfrac{23}{41}$ days, and C in $23\tfrac{7}{31}$ days.

2. It is required to find three numbers, such, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall together make 46; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, shall together make 35; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, shall together make $28\frac{1}{3}$. Ans. 12, 60, and 80.

3. A sum of money was divided among four persons, in such a manner, that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three, the share of the second $\frac{1}{3}$ the shares of the other three, and the share of the third $\frac{1}{4}$ the shares of the other three; and it was found that the share of the first exceeded that of the last by 14*l.*: What was the sum divided, and how much was each person's share?

Ans. The whole sum was 120*l.*; also the share of the first person was 40*l.*, of the second 30*l.*, of the third 24*l.*, and of the fourth 26*l.*

4. A person has 22*l.* 14*s.* in crowns, guineas, and moidores; and he finds that if he had as many guineas as crowns, and as many crowns as guineas, he should have 36*l.* 6*s.*; but if he had as many moidores as crowns, and as many crowns as moidores, he should have 45*l.* 16*s.* How many of each did he have?

Ans. 26 crowns, 9 guineas, and 5 moidores.

CHAPTER III.

ON RATIO, PROPORTION, AND PROGRESSION.

(63.) RATIO is the relation which one quantity bears to another of the same kind, with respect to magnitude.

ARITHMETICAL RATIO is that which expresses the *difference* of the quantities compared.

(64.) GEOMETRICAL RATIO expresses the *quotient* arising from the division of the quantities compared.

Thus, if a and b be compared, $b-a$ expresses their arithmetical ratio, and $\frac{b}{a}$ their geometrical ratio;* but, to prevent confusion, the term *ratio* is generally confined to the latter sense; and, instead of *arithmetical ratio*, the simple term *difference* is used; so that in what follows ratio always means geometrical ratio. Also, to avoid the too frequent repetition of the term, two dots are usually placed between the quantities to represent their ratio; thus, $a:b$ signifies the ratio of a to b; and a and b are called the terms of the ratio.

(65.) The first term in a ratio is called the *antecedent*, and the other the *consequent*.

(66.) In any number of ratios, if the antecedents and consequents be respectively multiplied together, the ratio of the products is said to be *compounded* of the preceding ratios: thus, in the following ratios, $a:b$, $c:d$, $e:f$, the product of the antecedents is ace, and that of the consequents bdf, and $ace:bdf$ is the *compound* ratio.

(67.) If the antecedents and consequents be respectively the same

* In expressing the geometrical ratio of two quantities, it matters not whether the second term be divided by the first, as is done here, or the first term by the second; but whichever way is fixed upon, that must be preserved. It is however usual, when ratios of different magnitudes are compared, to express them by the division of the first term by the second: thus, the ratio of 4 to 2 is said to be greater than that of 4 to 3, because $\frac{4}{2}$ is greater than $\frac{4}{3}$: but, in the investigation of properties, the way used in the text is rather preferable.

in each of the simple ratios, as $a : b$, $a : b$, $a : b$, &c. then the compound ratio is $a^2 : b^2$, or $a^3 : b^3$, &c. according to the number of simple ratios; in which case $a^2 : b^2$ is called the *duplicate* ratio of $a : b$, $a^3 : b^3$ the *triplicate* ratio, &c.; also, $\sqrt{a} : \sqrt{b}$ is called the *sub-duplicate*, $\sqrt[3]{a} : \sqrt[3]{b}$ the *sub-triplicate*, &c.

(68.) If each antecedent in the simple ratios be the same as the consequent in the preceding, as $a : b$, $b : c$, $c : d$, then the compound ratio, abc, &c. . bcd, &c. is evidently the same as $a : d$, because $\frac{bcd}{abc} = \frac{d}{a}$, d being supposed here to be the last consequent. The ratio also evidently continues the same if each term be either multiplied or divided by any quantity.

ON ARITHMETICAL PROPORTION AND PROGRESSION.

(69.) If there be four quantities, such, that the difference of the first and second is the same as that of the third and fourth, these quantities are said to be in arithmetical proportion.

(70.) If there be any number of quantities, such, that the difference of the first and second, of the second and third, of the third and fourth, &c. are all equal, these quantities are said to be in arithmetical progression, and the progression is said to be increasing or decreasing, according as the successive terms increase or decrease.

(71.) THEOREM 1. If four quantities be in arithmetical *proportion*, the sum of the extremes is equal to the sum of the means.

For let a, b, c, d, be in arithmetical proportion; then $b - a = d - c$; add $a + c$ to each side of this equation, and there results $b + c = a + d$.

THEOREM 2. If three quantities be in arithmetical progression. the sum of the extremes is equal to twice the mean.

For let a, b, c, be the three quantities; then $b - a = c - b$; add $b + a$ to each side of this equation, and there results $2b = a + c$.

THEOREM 3. In any series of quantities in arithmetical progression, the sum of the two extremes is equal to the sum of any

two terms equally distant from the extremes; or it is equal to twice the middle term, when the number of terms is odd.

For let a be the first term in the series, and d the common difference; then, if the series be increasing, it is a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c., in which, if the first and fifth be considered as extremes, we have

$$a + (a + 4d) = (a + d) + (a + 3d) = 2(a + 2d);$$

and the same may be shown for any greater number of terms; as also when the series is decreasing.

THEOREM 4. In any increasing arithmetical progression, the last term is equal to the first term *plus* the product of the common difference and number of terms *less* one; but if the progression be decreasing, then the last term is equal to the first term *minus* the same product.

Let a be the first term, and d the common difference; then the increasing series is a, $a+d$, $a+2d$, $a+3d$, &c., and the decreasing series is a, $a-d$, $a-2d$, $a-3d$, &c., where it is obvious that any term in the first series consists of the first term a, *plus* as many times d as are equal to the number of terms preceding the proposed term; and any term in the second series consists of the first term a, *minus* as many times d as are equal to the number of terms preceding; therefore the nth term of the former series is $a+(n-1)d$, and of the latter $a-(n-1)d$.

THEOREM 5. The sum of any series of quantities in arithmetical progression is equal to the sum of the extremes multiplied by half the number of terms.

Let $a + (a + d) + (a+2d) + (a + 3d) +$ &c. be the progression; then, if the number of terms be represented by n, the last term will be $a + (n-1)d$ (Theo. 4); and therefore, by reversing the terms, the same series may be written thus,

$$\{a+(n-1)d\} + \{a + (n-2)d\} + \{a + (n-3)d\} + \ldots\ldots$$
$$\{a + (n-n)d\},$$

and adding this series to its equal, as expressed above,

$$\{2a + (n-1)d\} + \{2a + (n-1)d\} + \{2a + (n-1)d\} + \ldots$$
$$\{2a + (n-1)d =$$

twice the sum of the progression; and as there must be n terms in

this last, as well as in the proposed series; and since each term is $2a + (n-1)d$, ... twice the sum $= \{n\,2a + (n-1)d\}$; and the sum $= \frac{1}{2}n\,\{2a + (n-1)d\}$; that is, the expression for the sum S is

$$S = \tfrac{1}{2}n\,\{2a + (n-1)d\}$$

$$\text{or } S = \tfrac{1}{2}n\,\{a + \text{last term}\}$$

and this formula is obviously quite sufficient to enable us to determine any one of the quantities a, d, n, S, or *last term*, when the others are given.

EXAMPLES.

1. Required the sum of 10 terms of the progression 1, 4, 7, 10, &c.

Here $a = 1$, $d = 3$, $n = 10$, and l (last term) $= a + 9d = 28$;

$$\therefore \frac{n(a+l)}{2} = \tfrac{290}{2} = 145, \text{ the sum required.}$$

2. The first term of an arithmetical progression is 14, and the sum of eight terms 28: What is the common difference?

Here the given quantities are $a = 14$, $n = 8$, and $S = 28$. Hence, making these substitutions in the general expression for S, we have

$$28 = 4\,\{28 + 7d\} = 112 + 28d$$

$$\therefore d = \frac{28 - 112}{28} = -3;$$

therefore the common difference is -3, and, consequently, the series is 14, 11, 8, 5, &c.

3. An arithmetical series consisting of six terms has 8 for the first term, and 23 for the last: Required the intermediate terms?

The expression for the last term l is $l = a + (n-1)d$, and in the question, a, l, and n are given to find d; that is, we have the equation,

$$23 = 8 + 5d \therefore d = \frac{23 - 8}{5} = 3;$$

hence, the first term being 8, and the common difference 3, the series must be 8, 11, 14, 17, 20, 23, where the four intermediate terms are exhibited. In this manner we may insert any proposed number of *arithmetical means* between two given numbers.

4. Required the sum of 100 terms of the series 1, 3, 5, 7, 9, &c.

Ans. 10000.

5. Required the sum of a decreasing arithmetical series, whose first term is 12, and the common difference of the terms $\frac{1}{2}$.

Ans. 150.

6. Required the sum of 25 terms of an arithmetical progression, whose first term is $\frac{1}{2}$, and the common increase of each term $\frac{1}{2}$.

Ans. $162\frac{1}{2}$.

7. Insert three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

The means are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

8. The first term of an arithmetical series is 1, the number of terms 23. What must the common difference be in order that the sum may be $149\frac{1}{2}$?

Ans. $\frac{1}{2}$.

GEOMETRICAL PROPORTION AND PROGRESSION.

PROPORTION.

(72.) If there be four quantities, such, that the ratio of the first and second is the same as that of the third and fourth, these quantities are said to be in *geometrical proportion:* Thus, if $\frac{b}{a} = \frac{d}{c}$, then a, b, c, d, are in geometrical proportion, and this proportion is represented thus, $a : b :: c : d$, which is read *a is to b as c to d*, or *as a is to b so is c to d.*

Hence, since if $\frac{b}{a} = \frac{d}{c}, \frac{b^n}{a^n} = \frac{d^n}{c^n}$, then $a^n : b^n :: c^n : d^n$; that is, if four quantities be proportional, then the same powers or roots of the four quantities are also in proportion.

(73.) Theorem 1. If four quantities be proportional, the product of the extremes is equal to that of the means.

Let a, b, c, d, be the proportionals, then $\frac{b}{a} = \frac{d}{c}$; multiply each side by ac, and there results $bc = ad$.

THEOREM 2. If the product of two quantities be equal to the product of two others, then a proportion may be formed of the four quantities.

Let $qr = ps$; then, dividing each side by rp, there results $\frac{q}{p} = \frac{s}{r}$, $\therefore p : q :: r : s$.

THEOREM 3. If four quantities be proportional, they are also proportional when taken inversely; that is, when the consequents are made antecedents, and the antecedents consequents.

For if $a : b :: c : d$, then (Theor. 1) $ad = bc$, and, consequently, (Theor. 2) $b : a :: d : c$.

THEOREM 4. If four quantities be proportional, they are proportional also when taken alternately; that is, the first is to the third as the second is to the fourth.

For if $a : b :: c : d$, then $\frac{b}{a} = \frac{d}{c}$; and multiplying by $\frac{c}{b}$, there results $\frac{c}{a} = \frac{d}{b}$, $\therefore a : c :: b : d$.

THEOREM 5. In any proportion, the first term is to the second *plus* or *minus* m times the first, as the third is to the fourth *plus* or *minus* m times the third.

Let $\frac{b}{a} = \frac{d}{c}$; then $\frac{b}{a} \pm m = \frac{d}{c} \pm m$, or $\frac{b \pm am}{a} = \frac{d \pm cm}{c}$; that is, $a : b \pm am :: c : d \pm cm$.

Corollary 1. Also, since $a : c :: b \pm am : d \pm cm$ (Theo. 4), $\therefore \frac{c}{a} = \frac{d \pm cm}{b \pm am}$; but $\frac{c}{a} = \frac{d}{b}$ $\therefore b : d :: b \pm am : d \pm cm$, and $b : b \pm am :: d : d \pm cm$; that is, the second term is to the second *plus* or *minus* m times the first, as the third is to the third *plus* or *minus* m times the fourth; also, since $\frac{d + cm}{b + am} = \frac{d - cm}{b - am}$, $\therefore b + am : d + cm :: b - am : d - cm$.

Cor. 2. And if m be taken $= 1$, we shall then have

$$a : b \pm a :: c : d \pm c, \text{ and } b : b \pm a :: d : d \pm c;$$

likewise,

$$b+a : d+c :: b-a : d-c, \text{ or } b+a : b-a :: d+c : d-c;$$

that is, the sum of the two first terms is to their difference as the sum of the two last to their difference.

THEOREM 6. In any number of proportions, if all the corresponding antecedents and consequents be respectively multiplied together, the resulting products will be in proportion.

$$\text{Let}\begin{cases}\frac{b}{a}=\frac{d}{c} \text{ or } a : b :: c : d\\ \frac{f}{e}=\frac{h}{g} \ldots e : f :: g : h\\ \frac{k}{i}=\frac{m}{l} \ldots i : k :: l : m\end{cases}$$

&c. &c. &c.

Then, multiplying the corresponding sides of the above equations together, we have

$$\frac{bfk, \&c.}{aei, \&c.}=\frac{dhm, \&c.}{cgl, \&c.}, \text{ or}$$

$$aei, \&c. : bfk, \&c. :: cgl, \&c. : dhm, \&c.$$

THEOREM 7. In any number of equal ratios, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let the ratios be $\frac{b}{a}=\frac{d}{c}=\frac{f}{e}=$ &c.: Put $\frac{b}{a}=q$;

then $b=aq$, $d=cq$, $f=eq$, &c., and, by adding these equations together, $b+d+f+\&c.=aq+cq+eq+\&c.$

$$=q(a+c+e+\&c.),$$

$$\therefore \frac{b+d+f+\&c.}{a+c+e+\&c.}=q=\frac{b}{a}=\frac{d}{c}=\&c., \text{ or}$$

$$a : b :: a+c+e+\&c. : b+d+f+\&c.$$

Cor. 1. Hence, in any number of proportions, where the ratio of the two first and two last terms are respectively the same in each, the sums of the corresponding terms are in proportion.

Cor. 2. Hence, also, in two proportions of this kind, if the terms of one be subtracted from the corresponding terms of the other, the remainders will be in proportion; since the results are the same as if each of the terms subtracted were multiplied by -1, and added.

Cor. 8. Therefore, in any number of proportions, having the same equality of the ratios, if the corresponding terms of some be added, and those of others subtracted, the final results will still be in proportion.

THEOREM 8. In any number of proportions, if the sum or difference of the first and second terms, as also of the third and fourth, be respectively the same in each, then the sums of the corresponding terms are also in proportion.

Let the proportions be

$$\begin{array}{ccccccc} a & : & b & :: & c & : & d \\ e & : & f & :: & g & : & h \\ i & : & k & :: & l & : & m \\ \&c. & & \&c. & & \&c. & & \&c. \end{array}$$

then, by Theorem 5, Cor. 2,

$$\begin{array}{ccccccc} b\pm a & : & d\pm c & :: & a & : & c \\ f\pm e & : & h\pm g & :: & e & : & g \\ k\pm i & : & m\pm l & :: & i & : & l \\ \&c. & & \&c. & & \&c. & & \end{array}$$

Now, if the first and second terms, in each of these proportions, be respectively the same, then the ratio of the third and fourth terms will be the same in all; but (Theor. 4),

$$\begin{array}{ccccccc} a & : & c & :: & b & : & d \\ e & : & g & :: & f & : & h \\ i & : & l & :: & k & : & m \\ \&c. & & \&c. & & \&c. & & \&c. \end{array}$$

∴ Theorem 7, Cor. 1,

$$a+e+i+\&c. : c+g+l+\&c. :: b+f+k+\&c. : d+h+m+\&c.$$

or (Theorem 4),

$$a+e+i+\&c. : b+f+k+\&c. :: c+g+l+\&c. : d+h+m+\&c.$$

Schol. Corollaries similar to the two last of Theorem 7 may evidently be deduced from this theorem.

PROGRESSION.

(74.) A GEOMETRICAL PROGRESSION is a series of quantities, such, that the quotient of any one of them, and that which immediately precedes, is constantly the same; that is, each is in the same constant ratio to the next following, throughout the series.

Thus, the following is a geometrical progression, in which a is the first term, r the constant ratio, and n the number of terms:

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6\ \ldots\ ar^{n-1}.$$

From the bare inspection of this series, the following properties are obvious:

1. If any two terms be taken as extremes, their product is equal to any two terms equally distant from them; or, if the number of terms be odd, the product of the extremes is equal to the square of the middle term; and hence a geometrical mean between two quantities is equal to the square root of their product.

2. The last term in any geometrical series is equal to the product of the first term, and that power of the ratio which is expressed by the number of terms, *minus* 1.

(75.) PROBLEM. To find the sum s of any number of terms in a geometrical series.

Let $s = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ ar^{n-1}$;

then multiplying each side by r, there results

$$sr = ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots\ ar^{n-1} + ar^n;$$

and subtracting the first equation from this, we have

$$sr - s = ar^n - a, \text{ whence } s = \frac{a(r^n - 1)}{r - 1};$$

or if the last term ar^{n-1} be represented by l, we have, by substitution,

$$s = \frac{rl - a}{r - 1}.$$

When, however, r is a proper fraction, and the series, which will then be a decreasing one, goes on to infinity, then the last term obviously becomes 0; and the expression for the sum is

$$s = \frac{a}{1 - r}.$$

Hence this rule:

(76.) Multiply the last term by the ratio, and divide the difference of this product and the first term by the difference between the ratio and unity; observing that in an infinite decreasing series the last term $= 0$.

EXAMPLES.

1. Required the sum of 9 terms of the series 1, 2, 4, 8, 16, &c.

Here $a=1$, $r=2$, and $n=9$, $\therefore ar^{n-1} = 2^8 = 256 =$ the last term; consequently, $\dfrac{256 \times 2 - 1}{2-1} = 511$, the sum required.

2. Required the sum of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c continued to infinity.

Here $a=1$, $r=\frac{1}{2}$, and $\therefore \dfrac{1}{1-\frac{1}{2}} = 2$, the sum required.

Given the first term 3, the last term 768, and the number of terms 9, to find the common ratio.

Here $a=3$, $l=768$, and $n=9$, and the general expression for the last term being ar^{n-1}, we have, in the present case,

$$768 = 3r^8 \therefore r = 256^{\frac{1}{8}} = 2;$$

hence the intermediate terms of the series are 6, 12, 24, 48, 96, 192, and 384: and in this way may any number of *geometric means* be interposed between any *two* given *extremes*.

4. Required the sum of 10 terms of the series 9, 27, 81, 243, &c.
Ans. 265716.

5. Required the sum of the series $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$, &c. continued to infinity.
Ans. $\frac{2}{3}$.

6. Required the sum of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c. continued to 10 terms.
Ans. $1\frac{9841}{19683}$.

7. Required the sum of 6 terms of the series $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} +$ &c.
Ans. $\frac{3367}{7168}$.

8. It is required to insert three geometric means between $\frac{1}{2}$ and $\frac{2}{9}$.
Ans. The means are $\frac{1}{2}\sqrt{\frac{2}{3}}$, $\frac{1}{3}$, and $\frac{1}{3}\sqrt{\frac{2}{3}}$.

9. Required the sum of the series $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} +$ &c. to infinity.

Ans. $\frac{x}{x-1}$.

10. Insert three geometric means between the extremes 4 and 324.

Ans. 12, 36, 108.

11. Suppose a body to move eternally in this manner, viz. 20 miles the first minute, 19 miles the second minute, $18\frac{1}{20}$ the third, and so in geometrical progression. Required the utmost distance it can reach.

Ans. 400 miles.

HARMONICAL PROPORTION.

(77.) Three quantities are said to be in harmonical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

(78.) And four quantities are in harmonical proportion, when the first has the same ratio to the fourth as the difference between the first and second has to the difference between the third and fourth.

Thus, the quantities a, b, c, are in harmonical proportion when $a : c :: a-b : b-c$; and, a, b, c, d, are in harmonical proportion when $a : d :: a-b : c-d$.

(79.) From these definitions it follows, that in three harmonical proportionals, a, b, c, any two being given, the third may be found;

For, since $a : c :: a-b : b-c$, $\therefore ab - ac = ac - bc$,

$$\text{or } ab + bc = 2ac;$$

$$\therefore b = \frac{2ac}{a+c};$$

that is, a harmonical mean between two quantities is equal to twice their product divided by their sum.

Also, $c = \dfrac{ab}{2a-b}$ = a third harmonical proportion to a and b.

(80.) In a similar manner, if any three out of four harmonical proportionals, a, b, c, d, be given, the other may be found; for since

$$a : d :: a-b : c-d, \therefore ac-ad = ad-bd,$$

and from this equation we get

$$b = \frac{2ad-ac}{d}; \; c = \frac{2ad-bd}{a}; \; d = \frac{ac}{2a-b}.$$

[REMARK.—The proportion $ab : cd :: a-b : c-d$ should be employed instead of that here given. It corresponds to four harmonics, a, b, c, d, of which the three first and also the three last will accord with definition 77. —— AM. PRS.]

(81.) QUESTIONS IN WHICH PROPORTION IS CONCERNED.

QUESTION I.

Find a number, such, that if 3, 8, and 17, be severally added thereto, the first sum shall be to the second as the second to the third

Let x be the number;

$$\text{then } x+3 : x+8 :: x+8 : x+17;$$

and by Cor. 2, Theor. 5, Art. 73, we have

$$x+3 : 5 :: x+8 : 9,$$

$\therefore$ (Theor. 1, Art. 73), $9x + 27 = 5x + 40$,

$$\text{or } 4x = 13$$

$$\therefore x = \tfrac{13}{4} = 3\tfrac{1}{4}, \text{ the number required.}$$

QUESTION II.

A person has British wine at 5*s*. per gallon, with which he wishes to mix spirits at 11*s*. per gallon, in such proportion, that by selling the mixture at 9*s*. a gallon, he may gain 35 per cent. What is the necessary proportion?

Let the proportion of the wine to the spirits be as $x : y$;

then $5x + 11y =$ prime cost of $x + y$ gallons,

and $9x + \;\;9y =$ selling price . . . . ,

$\therefore 4x - \;\;2y =$ profit . . . . ;

and by the question,

$$5x + 11y : 4x - 2y :: 100 : 35 :: 20 : 7 \text{ (Art. 72)};$$

$\therefore$ (Theo. 1, Art. 73), $80x - 40y = 35x + 77y$,

$$\text{or } 45x = 117y;$$

$$\therefore \;\; 5x = \;\;13y;$$

whence (Theor. 2,) $x : y \;\; : 13 : 5$;

$\therefore$ the mixture must be at the rate of 13 gallons of wine to 5 gallons of spirits.

QUESTION III.

A merchant having mixed a certain number of gallons of brandy and water, found, that if he had mixed 6 gallons more of each, there would have been 7 gallons of brandy to every 6 gallons of water; but, if he had mixed 6 gallons less of each, there would have been 6 gallons of brandy to every 5 gallons of water. How much of each did he mix?

Let x be the number of gallons of brandy,

y the number of gallons of water;

then, by the question, $\begin{cases} x + 6 : y + 6 :: 7 : 6 \\ x - 6 : y - 6 :: 6 : 5 \end{cases}$

$\therefore$ (Theorem 8), $\quad x \quad\quad y \quad :: 13 : 11$ (A);

also by the first proportion (Th. 5), $x - y : 1 :: x + 6 : 7$

and by the second .... $x - y : 1 :: x - 6 : 6$

$\therefore$ (Theor. 7), $x - 6 : 6 :: \;\; 2x \;\; : 13$

and (Theor. 1), $13x - 78 = 12x$,

$$\therefore x = 78;$$

also by substitution (A), $78 : y :: 13 : 11$,

$$\text{whence } 13y = 858$$

$$\therefore y = \;\;66;$$

consequently, the mixture consisted of 78 gallons of brandy, and 66 of water.

A much simpler solution to this question may be obtained as follows. Instead of representing the number of gallons of brandy by x, and the number of gallons of water by y, represent those quantities by $7x-6$, and $6x-6$, respectively, by which artifice the first condition in the question is at once fulfilled; so that we have only to express the second, viz.

$$7x-12 : 6x-12 :: 6 : 5$$

$$\text{or } x : 6x-12 :: 1 : 5,$$

$$\text{whence } 5x = 6x-12 \therefore x = 12;$$

$$\therefore 7x-6 = 78 \text{ gallons of brandy,}$$

$$\text{and } 6x-6 = 66 \text{ gallons of water.}$$

This neat solution is given in the American edition of this work, published by Mr. Ward, of Columbia College.

4. A corn-factor mixes wheat which cost 10*s.* a bushel with barley which cost 4*s.* a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11*s.* a bushel. What is the proportion?

Ans. There are 14 bushels of wheat to 9 of barley.

5. It is required to find a number, such, that the sum of its digits is to the number itself as 4 to 13; and if the digits be inverted, their difference will be to the number expressed as 2 to 31.

Ans. 39.

6. At a certain instant, between five and six o'clock, the hour and minute hands of a clock are exactly together. Required the time.

Ans. 27 minutes $16\frac{4}{11}$ seconds past 5.

7. Required two numbers, such, that their sum, difference, and product, may be as the numbers 3, 2, and 5, respectively.

Ans. 10 and 2.

8. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, and such, that if they be increased respectively by 6 and 5, they will be to each other as $\frac{3}{4}$ to $\frac{4}{5}$. What are the numbers?

Ans. 30 and 40.

9. A person has some choice brandy at 40*s*. 6*d*. per gallon, which he wishes to mix with other brandy at 36*s*. a gallon, in such proportion, that the compound may be worth 39*s*. 6*d*. a gallon. What must the proportion be?

Ans. 7 gallons of the best to 2 gallons of the other.

10. Find two numbers, such, that their sum, difference, and product, may be as the numbers s, d, and p, respectively.

Ans. $\frac{2p}{s+d}$ and $\frac{2p}{s-d}$.

11. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare? Ans. 300.

12. If three agents, A, B, C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d?

Ans. $d \div (\frac{a}{e} + \frac{b}{f} + \frac{c}{g})$.

13. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers?

Ans. 4, 24, 144, and 864.

14. A and B speculate in trade with different sums. A gains 150*l*., B loses 50*l*.; and now A's stock is to B's as 3 to 2; but, had A lost 50*l*. and B gained 100*l*., then A's stock would have been to B's as 5 to 9. What was the stock of each?

Ans. A's was 300*l*. and B's 350*l*.

CHAPTER IV.

ON QUADRATIC EQUATIONS.

ON QUADRATICS INVOLVING ONLY ONE UNKNOWN QUANTITY.

(82.) A QUADRATIC, as has been already defined, is an equation that contains the *second*, but no higher power of the unknown quantity or quantities.

(83.) Quadratic equations, involving but one unknown quantity, are therefore either of the form

$$ax^2 \pm b = \pm c,$$

$$\text{or } ax^2 \pm bx \pm c = \pm d;$$

and accordingly, as they come under the first or second of these forms, they are said to be PURE QUADRATICS, or ADFECTED QUADRATICS.

(84.) The solution of a Pure Quadratic is obviously a matter of but little difficulty; for, since it contains but one unknown term, $\pm ax^2$, if this term be made to stand by itself on one side of the equation, and the known terms on the other side, then the division of both sides by $\pm a$ will evidently produce an equation expressing the value of x^2; and the square root of this value must give that of x.

We shall therefore proceed to

ADFECTED QUADRATIC EQUATIONS.

(85.) Let $ax^2 \pm bx \pm c = \pm d$ be an adfected quadratic equation, then, by transposing and dividing by a, it becomes

$$x^2 \pm \frac{b}{a} x = \frac{\pm d \mp c}{a};$$

or, putting p for $\frac{b}{a}$, and $\pm q$ for $\frac{\pm d \mp c}{a}$, it is

$$x^2 \pm px = \pm q.$$

Add now the square of $\frac{1}{2}p$ to each side of this equation, and there results

$$x^2 \pm px + \tfrac{1}{4}p^2 = \pm q + \tfrac{1}{4}p^2,$$

where it is readily perceived that the first side is a complete square, viz. $(x \pm \frac{1}{2}p)^2$; consequently, if the square root of each side be extracted, we obtain $x \pm \frac{1}{2}p = \pm\sqrt{\pm q + \frac{1}{4}p^2}$ (the double sign $\pm$ being placed before the radical, because the square root of a quantity may be either $+$ or $-$)*; hence it appears that

$$x = +\sqrt{\pm q + \tfrac{1}{4}p^2} \mp \tfrac{1}{2}p, \text{ or } -\sqrt{\pm q + \tfrac{1}{4}p^2} \mp \tfrac{1}{2}p.\dagger$$

(86.) The above general values of x evidently include every possible case, from which separate formulæ for each distinct case are readily obtained, and are as follow:

In equations of the form,

$$x^2 + px = q,\ x = \begin{cases} +\sqrt{q + \frac{1}{4}p^2} - \frac{1}{2}p, \\ \text{or} -\sqrt{q + \frac{1}{4}p^2} - \frac{1}{2}p. \end{cases}$$

$$x^2 - px = q,\ x = \begin{cases} +\sqrt{q + \frac{1}{4}p^2} + \frac{1}{2}p. \\ \text{or} -\sqrt{q + \frac{1}{4}p^2} + \frac{1}{2}p. \end{cases}$$

* Conformably to the general practice, whenever the extraction of the square root is represented, the double sign $\pm$ is uniformly placed before the radical $\sqrt{\ }$, although it might be dispensed with. For, since the square root of a quantity is admitted to be either *plus* or *minus*, the symbol $\sqrt{\ }$ does virtually contain the double sign: its insertion, however, always reminds the student of this.

† These general expressions for x may also be obtained as follows: having reduced the proposed equation to the form $x^2 \pm px = \pm q$, as above, let us proceed actually to extract the square root of the first side. The process is as follows:

$$\begin{array}{r|l} & x^2 \pm px \;\big|\; x \pm \tfrac{1}{2}p \\ & x^2 \\ 2x \pm \tfrac{1}{2}p & \pm px \\ & \pm px + \tfrac{1}{4}p^2. \end{array}$$

It is obvious, from this, that the proposed expression is not a complete square, being indeed deficient by the quantity $\frac{1}{4}p^2$. If, therefore, we add this quantity to each side of the equation, and then extract the square root, we shall have, as above,

$$x \pm \tfrac{1}{2}p = \pm\sqrt{\pm q + \tfrac{1}{4}p^2}.$$

$$x^2 + px = -q, \; x = \begin{cases} +\sqrt{-q + \frac{1}{4}p^2} - \frac{1}{2}p, \\ \text{or} -\sqrt{-q + \frac{1}{4}p^2} - \frac{1}{2}p, \end{cases}$$

$$x^2 - px = -q, \; x = \begin{cases} +\sqrt{-q + \frac{1}{4}p^2} + \frac{1}{2}p, \\ \text{or} -\sqrt{-q + \frac{1}{4}p^2} + \frac{1}{2}p. \end{cases}$$

(87.) In the two last forms, if q be greater than $\frac{1}{4}p^2$, then $\sqrt{-q + \frac{1}{4}p^2}$ will be impossible, being the square root of a negative quantity; so if one value be impossible, the other is impossible also.

From the above formulæ* the value of the unknown, in any particular example, may be obtained by substitution; or the operations to be performed may be expressed at length as follow:

(88.) Bring all the unknown terms to one side of the equation, and the known terms to the other;

Divide each side of the equation by the coefficient of the unknown square, if it have a coefficient;

Add the square of half the coefficient of the simple unknown to each side of the equation, and the unknown side will then be a complete square;

Extract the square root of each side, and from the result the value of the unknown quantity is immediately deducible.

EXAMPLES.

1. Given $x^2 + 6x + 4 = 59$, to find the values of x.

By transposition, $x^2 + 6x = 55$,

and completing the square, $x^2 + 6x = 9 = 64$;

$\therefore$ extracting the root, $x + 3 = \pm\sqrt{64} = \pm 8$;

whence $x = 5$, or -11.

2. Given $2x^2 + 12x + 36 = 356$, to find the values of x.

By transposition, $2x^2 + 12x = 320$;

or dividing by 2, $x^2 + 6x = 160$,

and completing the square, $x^2 + 6x + 9 = 169$;

$\therefore$ extracting the root, $x + 3 = \pm\sqrt{169} = \pm 13$;

whence $x = 10$, or -16.

* Any general rule, expressed in algebraical language, is called a *formula*.

3. Given $10x^2 - 8x + 6 = 318$, to find the values of x.

By transposition, $10x^2 - 8x = 312$;
or dividing by 10, $x^2 - \frac{4}{5}x = \frac{156}{5}$,
and completing the square, $x^2 - \frac{4}{5}x + \frac{4}{25} = \frac{784}{25}$;
$\therefore$ extracting the root, $x - \frac{2}{5} = \pm\sqrt{\frac{784}{25}} = \pm\frac{28}{5}$;

$$\text{whence } x = 6, \text{ or } -5\tfrac{1}{5}.$$

4. Given $4x = \dfrac{36 - x}{x} + 46$, to find the values of x.

Clearing of fractions, $4x^2 = 36 - x + 46x = 36 + 45x$;
and by transposition, $4x^2 - 45x = 36$,

$$\text{or } x^2 - \tfrac{45}{4}x = 9,$$

and completing the square, $x^2 - \frac{45}{4}x + (\frac{45}{8})^2 = 9 + (\frac{45}{8})^2 = \frac{2601}{64}$;
$\therefore$ extracting the root, $x - \frac{45}{8} = \pm\sqrt{\frac{2601}{64}} = \pm\frac{51}{8}$;

$$\text{whence } x = 12, \text{ or } -\tfrac{3}{4}.$$

5. Given $5x - \dfrac{3x - 3}{x - 3} = 2x + \dfrac{3x - 6}{2}$, to find the values of x.

Clearing of fractions,

$$10x^2 - 36x + 6 = 4x^2 - 12x + 3x^2 - 15x + 18;$$

and by transposition, $3x^2 - 9x = 12$,

$$\text{or } x^2 - 3x = 4;$$

and completing the square, $x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}$;
$\therefore$ extracting the root, $x - \frac{3}{2} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$;

$$\text{whence } x = 4, \text{ or } -1.$$

6. Given $\dfrac{3}{x^2 - 3x} + \dfrac{6}{2x^2 + 8x} = \dfrac{27}{8x}$, to find the values of x.

Dividing by $\dfrac{3}{x}$, $\dfrac{1}{x - 3} + \dfrac{1}{x + 4} = \frac{9}{8}$;
and clearing the equation of fractions,

$$8x + 32 + 8x - 24 = 9x^2 + 9x - 108;$$

$\therefore$ by transposition, $116 = 9x^2 - 7x$, or rather $9x^2 - 7x = 116$,

$$\therefore x^2 - \tfrac{7}{9}x = \tfrac{116}{9};$$

and completing the square,

$$x^2 - \tfrac{7}{9}x + (\tfrac{7}{18})^2 = \tfrac{116}{9} + (\tfrac{7}{18})^2 = \tfrac{4225}{324};$$

$\therefore$ extracting the root, $x - \frac{7}{16} = \pm\sqrt{\frac{3249}{256}} = \pm\frac{57}{16}$;

whence $x = 4$, or $-\frac{25}{8}$.

7. Given $\sqrt{(4+x)(5-x)} = 2x - 10$, to find the values of x.

Squaring each side, $20 + x - x^2 = 4x^2 - 40x + 100$;

and by transposition, $5x^2 - 41x = -80$,

$$\therefore x^2 - \tfrac{41}{5}x = -16,$$

and completing the square,

$$x^2 - \tfrac{41}{5}x + (\tfrac{41}{10})^2 = -16 + (\tfrac{41}{10})^2 = \tfrac{81}{100};$$

$\therefore$ extracting the root, $x - \frac{41}{10} = \pm\sqrt{\frac{81}{100}} = \pm\frac{9}{10}$;

$$\therefore x = 5, \text{ or } \tfrac{32}{10} = 3\tfrac{1}{5}.$$

8. Given $\sqrt{3x - 5} = \dfrac{\sqrt{7x^2 + 36x}}{x}$, to find the values of x.

Squaring each side, $3x - 5 = \dfrac{7x^2 + 36x}{x^2} = \dfrac{7x + 36}{x}$;

and, multiplying by x, $3x^2 - 5x = 7x + 36$;

or, by transposition, $3x^2 - 12x = 36$,

$$\therefore x^2 - 4x = 12;$$

and completing the square, $x^2 - 4x + 4 = 16$;

$\therefore$ extracting the root, $x - 2 = \pm 4$;

whence $x = 6$, or -2.

9. Given $8x^2 + 6 = 7x + 171$, to find the values of x.

Ans. $x = 5$, or $-4\frac{1}{8}$.

10. Given $3x^2 = 42 - 5x$, to find the values of x.

Ans. $x = 3$, or $-4\frac{2}{3}$.

11. Given $4x - \dfrac{36 - x}{x} = 46$, to find the values of x.

Ans. $x = 12$, or $-\frac{3}{4}$.

12. Given $\dfrac{6(2x - 11)}{x - 3} + x - 2 = 24 - 3x$, to find the values of x.

Ans. $x = 6$, or $\frac{1}{2}$.

13. Given $\dfrac{120}{3x + 1} + \dfrac{90}{x} = 42$, to find the values of x.

Ans. $x = 3$, or $-\dfrac{5}{21}$.

14. Given $x^3 + (19 - x)^3 = 1843$, to find the values of x.

Ans. $x = 11$, or 8.

15. Given $325 + x : x :: 245 + x : 60$, to find the values of x.

Ans. $x = 75$, or -260.

16. Given $\{34 - 3(x-1)\}\dfrac{x}{2} = 57$, to find the values of x.

Ans. $x = 6$, or $6\frac{1}{3}$.

17. Given $\dfrac{10}{x} - \dfrac{14 - 2x}{x^2} = \frac{22}{9}$, to find the values of x.

Ans. $x = 3$, or $\frac{21}{11}$.

18. Given $\dfrac{y^3 - 10y^2 + 1}{y^2 - 6y + 9} = y - 3$, to find the values of y.

Ans. $y = 1$, or -28.

19. Given $\dfrac{6x^2 - 23x + 10}{9 - 2x} = -7x + 42$, to find the values of x.

Ans. $x = 11\frac{1}{2}$, or 4.

20. Given $x + \dfrac{\sqrt{x-3}}{2} = 8$, to find the values of x.

Ans. $x = 9\frac{1}{4}$, or 7.

21. Given $2x + \dfrac{x^3}{\sqrt{2x^4 - 3x^3}} = 2x(x+1)$, to find the values of x.

Ans. $x = \frac{3}{4} + \dfrac{\sqrt{11}}{4}$, or $\frac{3}{4} - \dfrac{\sqrt{11}}{4}$.

22. Given $\sqrt{4 + \sqrt{2x^3 + x^2}} = \dfrac{x+4}{2}$, to find the values of x.

Ans. $x = 12$, or 4.

23. Given $x^{\frac{3}{2}} + x^{\frac{5}{2}} = 6x^{\frac{1}{2}}$, to find the values of x.

Ans. $x = 2$, or -3.

24. Given $\sqrt[3]{x^3 - a^3} = x - b$, to find the values of x.

Ans. $x = \dfrac{b}{2} \pm \sqrt{\dfrac{4a^3 - b^3}{12b}}$.

25. Given $\dfrac{x+a}{x} + \dfrac{x}{x+a} = b$, to find the values of x.*

Ans. $x = \dfrac{a}{2}\left\{-1 \pm \sqrt{\dfrac{b+2}{b-2}}\right\}$.

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* By putting y for $\dfrac{x+a}{x}$, this equation will take the more simple form $y + \dfrac{1}{y} = b$.

26. Given $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$, to find the values of x.

Ans. $x=-a$, or $-b$.

27. Given $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x$, to find the values of x.

Ans. $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

28. Given $x-\dfrac{12+8\sqrt{x}}{x-5}=0$, to find the values of x.

Ans. $x=9$, or 4.

29. Given $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$, to find the values of x.

Ans. 9 or 4.

(89.) Every equation, containing only two unknown terms, may be reduced to a quadratic, provided the index of the unknown quantity in one term be double its index in the other; for, by putting y for the lowest power, or root of the unknown, y^2 will be the highest; so that the equation will become a quadratic.

EXAMPLES.

1. Given $x^n-2ax^{\frac{n}{2}}=b$, to find the values of x.

Completing the square,* $x^n-2ax^{\frac{n}{2}}+a^2=a^2+b$;

$$\therefore \text{extracting the root, } x^{\frac{n}{2}}-a=\pm\sqrt{a^2+b};$$

$$\therefore x=(a\pm\sqrt{a^2+b})^{\frac{2}{n}}.$$

2. Given $x+5=\sqrt{x+5}+6$, to find the values of x.

Putting $\sqrt{x+5}=y$, the equation becomes $y^2=y+6$;
or, by transposition, $y^2-y=6$,
and completing the square, $y^2-y+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

* The actual substitution of y for the unknown terms is not necessary, unless it have a compound form, in which case the substitution will often considerably contract the operation, and render it free from that complex appearance which it would otherwise exhibit.

$\therefore$ extracting the root, $y - \frac{1}{2} = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$,

$$\text{or } y = 3, \text{ or } -2;\text{*}$$

$$\therefore x + 5 (= y^2) = 9, \text{ or } 4,$$

$$\text{and } x = 4, \text{ or } -1.$$

3. Given $\sqrt{x+21} + \sqrt[4]{x+21} = 12$, to find the values of x.

Putting $\sqrt[4]{x+21} = y$, the equation becomes $y^2 + y = 12$;

and completing the square, $y^2 + y + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

$\therefore$ extracting the root, $y + \frac{1}{2} = \pm\frac{7}{2}$;

$$\therefore y = 3, \text{ or } -4;$$

and $x + 21 (= y^4) = 81$, or 256;

$$\therefore x = 60, \text{ or } 235.$$

4. Given $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3$, to find the values of x.

Adding 9 to each side, $2x^2+3x+9-5\sqrt{2x^2+3x+9} = 6$;

and putting $\sqrt{2x^2+3x+9} = y$, the equation becomes

$$y^2 - 5y = 6;$$

$\therefore$ completing the square, $y^2 - 5y + \frac{25}{4} = 6 + \frac{25}{4} = \frac{49}{4}$;

and extracting the root, $y - \frac{5}{2} = \pm\frac{7}{2}$;

$$\therefore y = 6, \text{ or } -1:$$

and taking $y = 6$, $2x^2+3x+9 (= y^2) = 36$,

$$\text{or } x^2 + \frac{3}{2}x = \frac{27}{2};$$

and completing the square, $x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{27}{2} + \frac{9}{16} = \frac{225}{16}$;

$\therefore$ extracting the root, $x + \frac{3}{4} = \pm\frac{15}{4}$;

$$\text{whence } x = 3, \text{ or } -\frac{9}{2}:$$

or, taking $y = -1$, $2x^2 + 3x + 9 = 1$;

$$\text{or } x^2 + \frac{3}{2}x = -4;$$

and completing the square, $x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{-55}{16}$;

$\therefore$ extracting the root, $x + \frac{3}{4} = \pm\frac{\sqrt{-55}}{4}$:

* [The negative values of y, in this and in the two following examples, should have been omitted by the author.——AM. PRS.]

$$\text{whence } x = \frac{-3 \pm \sqrt{-55}}{4}.$$

5. Given $(2x+6)^{\frac{1}{2}} + (2x+6)^{\frac{1}{4}} = 6$, to find the values of x.

Ans. $x = 5$.

6. Given $\dfrac{1}{(2x-4)^2} = \frac{1}{8} + \dfrac{2}{(2x-4)^4}$, to find the values of x.

Ans. $x = 3$, or 1.

7. Given $3x^{\frac{4}{3}} - \dfrac{5x^{\frac{2}{3}}}{2} + 592 = 0$, to find the values of x.

Ans. $x = 8$, or $-(\frac{74}{3})^{\frac{3}{2}}$.

8. Given $(x+12)^{\frac{1}{2}} = 6 - (x+12)^{\frac{1}{4}}$, to find the values of x.

Ans. $x = 4$.

9. Given $x = \dfrac{\sqrt{x^4 - a^4}}{a}$, to find the values of x.

Ans. $x = \pm a \sqrt{\dfrac{1 \pm \sqrt{5}}{2}}$.

10. Given $x^{\frac{5}{2}} - x = 56x^{-\frac{1}{2}}$, to find the values of x.

Ans. $x = 4$, or $\sqrt[3]{49}$.

11. Given $3x^2 + x^{\frac{7}{6}} - 3104x^{\frac{1}{3}} = 0$, to find the values of x.

Ans. $x = 64$, or $(\frac{97}{-3})^{\frac{6}{5}}$.

12. Given $[(2x+1)^2 + x]^2 - x = 90 + (2x+1)^2$, to find the values of x.

Ans. $x = \left\{ \begin{array}{c} 1 \\ \text{or } -2\frac{1}{2} \end{array} \right\}$, or $x = -\frac{5}{8} \pm \dfrac{\sqrt{-135}}{8}$.

ANOTHER METHOD OF SOLVING QUADRATICS.

(90.) Let the equation $ax^2 \pm bx = c$ be multiplied by $4a$, then $4a^2x^2 \pm 4abx = 4ac$, and if b^2 be added to each side, the equation becomes $4a^2x^2 \pm 4abx + b^2 = 4ac + b^2$; now the first side is evidently a square, $= (2ax \pm b)^2$, whence

$$2ax \pm b = \pm \sqrt{4ac + b^2}, \therefore x = \frac{\pm \sqrt{4ac + b^2} \mp b}{2a}.$$

Hence the following rule:

(91.) Having transposed the unknown terms to one side of the equation, and the known terms to the other, multiply each side by 4 times the coefficient of the unknown square.

Add the square of the coefficient of the simple power of the unknown, in the proposed equation, to both sides, and the unknown side will then be a complete square.

Extract the root, and the value of the unknown quantity is obtained as before.*

EXAMPLES.

1. Given $3x^2+5x-8=34$, to find the values of x.

By transposition, $3x^2+5x=42$;

and multiplying by 4×3, or 12, $36x^2+60x=504$;

and completing the square, by adding 5^2,

$$36x^2+60x+25=529;$$

$\therefore$ extracting the root, $6x+5=\pm 23$;

$$\text{whence } x=\frac{\pm 23-5}{6}=3, \text{ or } -4\tfrac{2}{3}.$$

2. Given $x^2+6x+4=22-x$, to find the values of x.

By transposition, $x^2+7x=18$;

and multiplying by 4, $4x^2+28x=72$;

$\therefore$ completing the square, $4x^2+28x+49=121$;

and extracting the root, $2x+7=\pm 11$;

$$\text{whence } x=\frac{\pm 11-7}{2}=2, \text{ or } -9.$$

3. Given $8x^2-7x+6=171$, to find the values of x.

By transposition, $8x^2-7x=165$;

and multiplying by 4×8, or 32, $256x^2-224x=5280$;

$\therefore$ completing the square, $256x^2-224x+49=5329$;

and extracting the root, $16x-7=\pm 73$;

* This method is taken from the Bija Ganita, a Hindoo treatise on Algebra, translated from a Persian copy by Mr. Strachey. For an account of this curious work, see Dr. Hutton's Tracts, vol. ii. page 162.

$$\therefore x = \frac{\pm 73 + 7}{16} = 5, \text{ or } -\tfrac{33}{8}.$$

4. Given $\sqrt{x + 12} = \dfrac{12}{\sqrt{x + 5}}$, to find the values of x.

Squaring each side, $x + 12 = \dfrac{144}{x + 5}$;

and multiplying by $x + 5$, $x^2 + 17x + 60 = 144$;

$\therefore$ by transposition, $x^2 + 17x = 84$;

and multiplying by 4, $4x^2 + 68x = 336$;

$\therefore$ completing the square, $4x^2 + 68x + 289 = 625$;

and extracting the root, $2x + 17 = \pm 25$:

$$\therefore x = \frac{\pm 25 - 17}{2} = 4.$$

(92.) It will have been perceived, from the preceding solutions, that in equations of the form $ax^2 \pm bx = c$, where a is a small number, and in those of the form $x^2 \pm px = q$, where p is odd; this second method is more commodious than the former. One great advantage is, that it does not introduce *fractions* into the operation. It will be unnecessary to add any more examples illustrative of this method, as those already given (Art. 88) will also suffice for this purpose.

We may however here point out an obvious simplification in the process, which it would be worth while to attend to in practice. It appears, from the foregoing general formula, that any quadratic $ax^2 \pm bx = c$, is reducible to the simple equation

$$2ax \pm b = \pm \sqrt{4ac + b^2},$$

which reduced form may in practice be written down at once from the proposed equation, without the aid of any intermediate steps: for, if we double the first coefficient in the proposed equation, we shall have the proper coefficient for x in the reduced equation; and if to the first term thus found we connect with its proper sign the second coefficient in the proposed, the first side of the reduced equation will be formed.* The second side will be had by multiplying

* The more advanced student will at once see that this first side is the side of the *limiting equation* to the proposed.

the absolute term (that is, the second side,) in the proposed by four times the first coefficient, adding to the result the square of the second coefficient, and covering the whole by the sign of the square root. As an illustration, take Example 1, Art. 91, which, after transposition, is

$$3x^2 + 5x = 42;$$

then, forming each side of the reduced equation as above directed, we get immediately

$$6x + 5 = \pm\sqrt{12 \times 42 + 25},$$

$$\text{that is, } 6x + 5 = \pm 23;$$

$$\therefore x = 3, \text{ or } -4\tfrac{2}{3}.$$

The second example, after transposition, is

$$x^2 + 7x = 18,$$

$$\therefore 2x + 7 = \pm\sqrt{4 \times 18 + 49} = \pm 11;$$

$$\therefore x = 2, \text{ or } -9.$$

When the coefficients and absolute term in a quadratic equation are very large numbers, the solution may be more expeditiously obtained by the method explained in *The Chapter on the General Theory and Solution of Equations of all Degrees*, which forms a supplement to the present volume.

(93.) QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

QUESTION I.

It is required to find two numbers, whose difference shall be 12, and product 64.

Let x be the less number;

then $x + 12$ is the greater:

also by the question, $x(x + 12) = 64$,

that is, $x^2 + 12x = 64$;

$\therefore$ completing the square, $x^2 + 12x + 36 = 100$;

and extracting the root, $x + 6 = \pm 10$;

$$\therefore x = \pm 10 - 6 = 4, \text{ or } -16:$$

hence the numbers are either 4 and 16, or -16 and -4.

QUESTION II.

Having sold a commodity for 56*l.*, I gained as much per cent. as the whole cost me. How much then did it cost?

Suppose it cost x pounds;

then the gain was $56 - x$;

and by the question, $100 : x :: x : 56 - x$;

$$\therefore x^2 = 5600 - 100x;$$

or, by transposition, $x^2 + 100x = 5600$;

and completing the square, $x^2 + 100x + 2500 = 8100$;

$\therefore$ extracting the root, $x + 50 = \pm 90$;

$$\text{whence } x = 40, \text{ or } -140;$$

$\therefore$ the commodity cost 40*l.*: the other value of x is inadmissible.

QUESTION III.

A company at a tavern had 8*l.* 15*s.* to pay; but, before the bill was paid, two of them went away, when those who remained had, in consequence, 10*s.* each more to pay. How many persons were in company at first?

Let x be the number;

then, $\frac{175}{x}$ is the number of shillings each had to pay at first;

and by the question, $\frac{175}{x} + 10$ is the number each had to pay after two had gone:

$$\therefore \left(\frac{175}{x} + 10\right)(x - 2) = 175;$$

that is, $\frac{175x - 350}{x} + 10x - 20 = 175$;

$$\therefore 175x - 350 + 10x^2 = 195x;$$

$$\text{or } 10x^2 - 20x = 350;$$

$$\therefore x^2 - 2x = 35;$$

and completing the square, $x^2 - 2x + 1 = 36$;

$\therefore$ extracting the root, $x - 1 = \pm 6$;

whence, $x = 7$, or -5:

$\therefore$ there were seven persons at first.

QUESTION IV.

A person travels from a certain place at the rate of one mile the first day, two the second, and so on; and, in six days after, another sets out from the same place, in order to overtake him, and travels uniformly at the rate of fifteen miles a day. In how many days will they be together?

Let x be the number of days;

Then the first will have travelled $x + 6$ days;

and (Art. 30, Theo. 5, chap. 2), $(x+7) \cdot \frac{x+6}{2}$ is the distance gone:

also $15x$ is the distance the second travels;

$$\therefore (x+7) \cdot \frac{x+6}{2} = 15x;$$

$$\text{and, } \therefore x^2 + 13x + 42 = 30x;$$

or, by transposition, $x^2 - 17x = -42$;

and completing the square (Art. 91), $4x^2 - 68x + 289 = 121$;

$\therefore$ extracting the root, $2x - 17 = \pm 11$;

$$\text{whence } x = \frac{\pm 11 + 17}{2} = 14, \text{ or } 3;$$

hence it appears, that they will be together 3 days after the second sets out, who will then overtake the first, and be overtaken by him again in 11 days after, or 14 from the time of the second setting out.

QUESTION V.

A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*, and finds that he has sold 3 dozen more of sherry for 10*l.* than he has of claret for 6*l.* Required the price of each.

Let x be the price of a dozen of sherry in pounds;

then $\frac{10}{x}$ = the no. of doz. of sherry for 10*l.*

16. What two numbers are those, whose sum multiplied by their product, is equal to 12 times the difference of their squares; and which are to each other in the ratio of 2 to 3?

Ans. 4 and 6.

17. A person being asked his age, said, "The number representing my age is equal to 10 times the sum of its two digits; and the square of the left hand digit is equal to $\frac{1}{4}$ of my age." Required the person's age.

Ans. 20.

18. Two partners, *A* and *B*, gained 18*l.* by trade. *A*'s money was in trade 12 months, and he received for his principal and gain 26*l.*: also *B*'s money, which was 30*l.*, was in trade 16 months. What money did *A* commence with?

Ans. 20*l.*

19. The joint stock of two partners, *A* and *B*, was 416*l.* *A*'s money was in trade 9 months, and *B*'s 6 months: when they shared stock and gain, *A* received 228*l.*, and *B* 252*l.* What was each man's stock?

Ans. *A*'s stock was 192*l.*, and *B*'s 224*l.*

20. Required the dimensions of a rectangular field, whose length may exceed its breadth by 16 yards, and whose surface may measure 960 square yards.

Ans. Length 40 yards, breadth 24 yards.

21. The plate of a looking-glass is 18 inches by 12, and it is to be surrounded by a plain frame of uniform width, and of surface equal to that of the glass. Required the width of the frame.

Ans. 3 inches.

22. The difference between the hypotenuse and base of a right-angled triangle is 6, and the difference between the hypotenuse and perpendicular is 3. What are the sides?

Ans. 15, 9, and 12.

23. There are three numbers in geometrical proportion; the sum of the first and second is 15, and the difference of the second and third is 36. What are the numbers?

Ans. 3, 12, and 48.

24. It is found, by experiment, that bodies in falling to the earth pass through about $16\frac{1}{12}$ feet in the first second of their motion, and it is known that the spaces passed through from the commencement of motion are as the squares of the intervals elapsed. Suppose, then, that a drop of rain be observed to fall through 595 feet during the last second of its descent, required the height from which it fell?

Ans. $5804\frac{3}{4}$ feet nearly.

ON QUADRATICS INVOLVING TWO UNKNOWN QUANTITIES.

(94.) Equations containing two unknown quantities, in the form of quadratics, cannot be solved, *generally*, by any of the preceding rules, as their solution, in many instances, can only be obtained by means of equations of higher degrees: in several cases, however, their solution may be effected by help of the foregoing methods. These cases we shall now explain.

(95.) *When one of the given Equations is in the form of a Simple Equation.*

Find the value of one of the unknown quantities in the simple equation, in terms of the other and known quantities, and substitute this value for that quantity in the other equation, which will then be a quadratic containing only one unknown quantity.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + y = 10 \\ 2x^2 - xy + 3y^2 = 54 \end{array}\right\}$, to find the values of x and y.

From the first equation,

$$x = \frac{10-y}{2}, \text{ whence } 2x^2 = \frac{100-20y+y^2}{2},$$

$$\text{and } xy = \frac{10y-y^2}{2};$$

$\therefore$ the second equation becomes, by substitution,

$$\frac{100-20y+y^2}{2} - \frac{10y-y^2}{2} + 3y^2 = 54;$$

and clearing this equation of fractions,

$$100-20y+y^2-10y+y^2+6y^2 = 108;$$

and by transposition,

$$8y^2 - 30y = 8, \text{ or } y^2 - \tfrac{15}{4}y = 1$$

$\therefore$ completing the square,

$$y^2 - \tfrac{15}{4}y + \tfrac{225}{64} = \tfrac{289}{64};$$

and extracting the root,

$$y - \tfrac{15}{8} = \pm \tfrac{17}{8};$$

$$\therefore\ y = 4, \text{ or } -\tfrac{1}{4},$$

$$\text{and } x = 3, \text{ or } \tfrac{41}{8}.$$

2. Given $\left\{ \begin{aligned} \frac{4x + 2y}{3} &= 6 \\ 5xy &= 50 \end{aligned} \right\}$, to find the values of x and y.

From the first equation,

$$x = \frac{9 - y}{2},$$

$$\therefore\ 5xy = \frac{45y - 5y^2}{2} = 50;$$

hence $45y - 5y^2 = 100$, or $5y^2 - 45y = -100$;

$$\therefore\ y^2 - 9y = -20;$$

and completing the square,

$$4y^2 - 36y + 81 = 1;$$

$\therefore$ extracting the root,

$$2y - 9 = \pm 1;$$

$$\text{whence } y = \frac{\pm 1 + 9}{2} = 5, \text{ or } 4;$$

$$\text{and } x = \frac{9 - y}{2} = 2, \text{ or } 2\tfrac{1}{2}.$$

3. Given $\left\{ \begin{aligned} \frac{10x + y}{xy} &= 3 \\ y - x &= 2 \end{aligned} \right\}$, to find the values of x and y.

From the second equation,

$$y = x + 2,$$

and from the first,

$$10x + y = 3xy,$$

Substituting in this the value of y just found,

$$10x + x + 2 = 3x^2 + 6x,$$

$\therefore$ by transposition,

$$3x^2 - 5x = 2,$$

hence (page 111)

$$6x - 5 = \pm\sqrt{24 + 25}$$

$$\therefore x = \frac{5 \pm 7}{6} = 2, \text{ or } -\tfrac{1}{3}.$$

consequently, $y = x + 2 = 4$, or $1\tfrac{2}{3}$.

4. Given $\left\{ \begin{array}{l} 4xy = 96 - x^2y^2 \\ x+y = \;\; 6 \end{array} \right\}$, to find the values of x and y.

From the first equation, by transposition,

$$x^2y^2 + 4xy = 96;$$

and substituting for x, its value, $6 - y$, as obtained from the second, we have

$$(6 - y)^2y^2 + 4(6 - y)\,y = 96,$$

or putting $(6 - y)\,y = z$,

$$z^2 + 4z = 96;$$

$\therefore$ completing the square,

$$z^2 + 4z + 4 = 100;$$

and extracting,

$$z + 2 = \pm 10;$$

$$\therefore z, \text{ or } 6y - y^2, = 8, \text{ or } -12;$$

$$\therefore y^2 - 6y, = -8, \text{ or } 12;$$

and completing the square,

$$y^2 - 6y + 9 = 1, \text{ or } 21;$$

$\therefore$ extracting the root,

$$y - 3 = \pm 1, \text{ or } \pm\sqrt{21}:$$

whence $y = 4$, or 2; or $3 \pm \sqrt{21}$; and $x\,(= 6 - y) = 2$, or 4; or $3 \mp \sqrt{21}$.

5. Given $\left\{ \begin{array}{r} x^n + y^n = a \\ xy = b \end{array} \right\}$, to find the values of x and y.

By squaring the first equation,

$$x^{2n} + 2x^ny^n + y^{2n} = a^2$$

4 times the nth power of the second, gives

$$4x^ny^n \quad = 4b^n.$$

By subtraction,

$$x^{2n} - 2x^ny^n + y^{2n} \quad = a^2 - 4b^n.$$

Extracting the root,

$$x^n - y^n = \sqrt{a^2 - 4b^n}.$$

Therefore, by adding and subtracting this from the first of the given equations, and then taking the nth root, we have

$$x = \{\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - 4b^n}\}^{\frac{1}{n}}$$

$$y = \{\tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - 4b^n}\}^{\frac{1}{n}}$$

6. Given $\left\{\begin{array}{l} \dfrac{x}{y^2} = 2 \\ \tfrac{1}{3}(x-y) = 5 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 18 \text{ or } 12\tfrac{1}{2} \\ y = \ \ 3 \text{ or } -2\tfrac{1}{2} \end{array}\right.$

7. Given $\left\{\begin{array}{r} x + 4y = 14 \\ y^2 - 2y + 4x = 11 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2 \text{ or} -46 \\ y = 3 \text{ or } \ \ 15 \end{array}\right.$

8. Given $\left\{\begin{array}{l} 2x + y = 22 \\ \dfrac{xy}{2} + y^2 = 60 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 8, \text{ or } \ \ 17\tfrac{2}{3}, \\ y = 6, \text{ or} -13\tfrac{1}{3}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x = 15 + y \\ y^3 = \dfrac{xy}{2} \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 18, \text{ or } \ 12\tfrac{1}{2}, \\ y = \ \ 3, \text{ or} -2\tfrac{1}{2}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} x + 3y = 16 \\ 3x^2 + 2xy - y^2 = -12 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 1, \text{ or} -7\tfrac{2}{5}, \\ y = 5, \text{ or } \ \ 7\tfrac{4}{5}. \end{array}\right.$

11. Given $\left\{\begin{array}{l} x+y:x-y::13:5 \\ x+y^2=25 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=9, \text{ or } -14\frac{1}{16}, \\ y=4, \text{ or } -\ \ 6\frac{1}{4}. \end{array}\right.$

12. Given $\left\{\begin{array}{l} \dfrac{x^2}{y^2}+\dfrac{4x}{y}=\dfrac{85}{9} \\ x-y=2 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=5 \text{ or } \ \ 1\frac{7}{10}, \\ y=3 \text{ or } -\frac{3}{10}. \end{array}\right.$

(96.) If the simple equation consist of the sum or product of the unknown quantities, and the other equation of either the sum of their squares, the sum of their cubes, the sum of their fourth powers, &c. then the solution is obtained by employing a mode somewhat different from that above given, as in the following general examples:

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y=a \\ x^2+y^2=b \end{array}\right\}$, to find the values of x and y.

By squaring the first equation,

$$x^2+2xy+y^2=a^2;$$

and subtracting the second, $x^2 \qquad + y^2 = b,$

we have............... $2xy \quad = a^2-b:$

Also, subtracting this from the second equation,

$$x^2-2xy+y^2=2b-a^2;$$

and, since the first side of this equation is $(x-y)^2$, we have, by extracting the root,

$$x-y=\pm\sqrt{2b-a^2};$$

but $x+y=a$; therefore

$$(x+y)+(x-y)=2x=a\pm\sqrt{2b-a^2},$$

$$\text{or } x=\frac{a\pm\sqrt{2b-a^2}}{2};$$

and $(x+y)-(x-y)=2y=a\mp\sqrt{2b-a^2}$,

$$\text{or } y=\frac{a\mp\sqrt{2b-a^2}}{2}$$

Or thus:

Put $x = s + z$, and $y = s - z$ · then $x + y = 2s$, or $s = \frac{a}{2}$;

$$\therefore\ x^2 = s^2 + 2sz + z^2,$$
$$\text{and } y^2 = s^2 - 2sz + z^2;$$

$\therefore$ by addition, $\quad x^2 + y^2 = 2s^2 + 2z^2 = b$:

$$\text{whence } z^2 = \frac{b - 2s^2}{2}, \text{ and } \therefore\ z = \pm\sqrt{\frac{b-2s^2}{2}},$$

$$\text{and } x = s + z = s \pm \sqrt{\frac{b-2s^2}{2}};$$

$$\text{also } y = s - z = s \mp \sqrt{\frac{b-2s^2}{2}};$$

$\therefore$ by restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{\frac{b - \frac{a^2}{2}}{2}} = \frac{a \pm \sqrt{2b - a^2}}{2},$$

$$\text{and } y = \frac{a}{2} \mp \sqrt{\frac{b - \frac{a^2}{2}}{2}} = \frac{a \mp \sqrt{2b - a^2}}{2}, \text{ as before.}$$

2. Given $\left\{\begin{matrix} x + y = a \\ x^3 + y^3 = c \end{matrix}\right\}$, to find the values of x and y.

By cubing the first equation,

$$x^3 + 3x^2y + 3xy^2 + y^3 = a^3;$$

subtracting the second, $\quad x^3 + y^3 \quad = c,$

we have...... $\quad 3x^2y + 3xy^2 \quad = a^3 - c,$

or $3(x+y)xy = 3axy = a^3 - c$; $\therefore\ x = \dfrac{a^3 - c}{3ay}$;

and, by substitution,

$$\frac{a^3 - c}{3ay} + y = a, \text{ or } a^3 - c + 3ay^2 = 3a^2y;$$

$\therefore$ by transposing, and dividing by $3a$,

$$y^2 - ay = \frac{c - a^3}{3a};$$

and completing the square,

$$y^2 - ay + \frac{a^2}{4} = \frac{c - a^3}{3a} + \frac{a^2}{4} = \frac{4c - a^3}{12a};$$

$\therefore$ extracting the root,

$$y - \frac{a}{2} = \pm\sqrt{\frac{4c - a^3}{12a}}, \text{ and } y = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}};$$

$$\therefore\ x = a - y = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}}.$$

Or, thus:

Putting $x = s + z$, and $y = s - z$, as in the preceding example, we have

$$x^3 = s^3 + 3s^2z + 3sz^2 + z^3;$$
$$y^3 = s^3 - 3s^2z + 3sz^2 - z^3;$$

$$\therefore \text{ by addition, } x^3 + y^3 = 2s^3 + 6sz^2 = c;$$

$$\text{whence } z^2 = \frac{c - 2s^3}{6s}, \text{ and } z = \pm\sqrt{\frac{c - 2s^3}{6s}};$$

$$\therefore\ x = s \pm \sqrt{\frac{c - 2s^3}{6s}}; \text{ and } y = s \mp \sqrt{\frac{c - 2s^3}{6s}};$$

$\therefore$ by restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{\frac{c - \frac{a^3}{4}}{3a}} = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}};$$

$$\text{and } \therefore\ y = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}}, \text{ as before.}$$

6. Given $\left\{ \begin{matrix} x + y = a \\ x^4 + y^4 = d \end{matrix} \right\}$, to find the values of x and y.

By involving the first equation to the fourth power,

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=a^4;$$

and subtracting the second, $x^4+y^4 \qquad = d;$

there results . . . . $4x^3y+6x^2y^2+4xy^3 \qquad = a^4-d;$

$\therefore$ dividing by xy . . $4x^2+6xy+4y^2 \qquad = \dfrac{a^4-d}{xy}:$

$$\text{Now. } 4(x+y)^2 = 4x^2+8xy+4y^2 = 4a^2;$$

hence, by subtraction, $2xy = 4a^2 - \dfrac{a^4-d}{xy}$,

$$\text{or } 2x^2y^2 = 4a^2xy - a^4 + d;$$

$\therefore$ by transposition and division,

$$x^2y^2 - 2a^2xy = \frac{d-a^4}{2};$$

and completing the square,

$$x^2y^2 - 2a^2xy + a^4 = \frac{d-a^4}{2} + a^4 = \frac{d+a^4}{2};$$

$\therefore$ extracting the root,

$$xy - a^2 = \pm\sqrt{\frac{d+a^4}{2}}, \text{ and } xy = a^2 \pm \sqrt{\frac{d+a^4}{2}};$$

and putting, for simplicity's sake, $a^2 \pm \sqrt{\dfrac{d+a^4}{2}} = m$,

we have, by substitution,

$$\frac{m}{y} + y = a \text{ or } m + y^2 = ay, \text{ or } y^2 - ay = -m;$$

$\therefore$ completing the square and extracting the root,

$$y - \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - m}:$$

$$\text{whence } y = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - m} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - a^2 \mp \sqrt{\frac{d+a^4}{2}}}$$

$$= \frac{a}{2} \pm \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d+a^4}{2}}}$$

$$\text{and } \therefore x = a - y = \frac{a}{2} \mp \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d+a^4}{2}}}.$$

Or, thus:

Putting $x = s + z$, and $y = s - z$, as in the preceding examples, we have

$$x^4 = s^4 + 4s^3z + 6s^2z^2 + 4sz^3 + z^4,$$
$$y^4 = s^4 - 4s^3z + 6s^2z^2 - 4sz^3 + z^4;$$

$\therefore$ by addition, $x^4 + y^4 = 2s^4 + 12s^2z^2 + 2z^4 \quad = d;$

and dividing by 2, . . . $s^4 + 6s^2z^2 + z^4 = \frac{d}{2};$

$$\therefore z^4 + 6s^2z^2 = \frac{d}{2} - s^4;$$

and completing the square, and extracting the root,

$$z^2 + 3s^2 = \pm\sqrt{\frac{d}{2} + 8s^4};$$

$$\therefore z = \pm\sqrt{-3s^2 \pm \sqrt{\frac{d}{2} + 8s^4}};$$

consequently, $x = s \pm \sqrt{-3s^2 \pm \sqrt{\frac{d}{2} + 8s^4}},$

and $y = s \mp \sqrt{-3s^2 \pm \sqrt{\frac{d}{2} + 8s^4}};$

or restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{-\frac{3a^2}{4} \pm \sqrt{\frac{d + a^4}{2}}},$$

and $y = \frac{a}{2} \mp \sqrt{-\frac{3a^2}{4} \pm \sqrt{\frac{d + a^4}{2}}}$, as before.

4. Given $\left\{ \begin{array}{l} x + y = a \\ x^5 + y^5 = e \end{array} \right\}$, to find the values of x and y.

By involving the first equation to the fifth power,

$$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = a^5;$$

and subtracting the second, $x^5 + y^5 \quad = e;$

we have . . $5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 \quad = a^5 - e;$

and dividing by $5xy$, $x^3 + 2x^2y + 2xy^2 + y^3 = \frac{a^5 - e}{5xy}$:

But $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = a^3$;

$\therefore$ by subtraction . . $x^2y + xy^2 = a^3 - \frac{a^5 - e}{5xy}$;

$$\text{or } (x+y)\,xy = axy = a^3 - \frac{a^5 - e}{5xy};$$

$\therefore$ multiplying by $5xy$, and transposing, we have

$$5ax^2y^2 - 5a^3xy = e - a^5,$$

$$\text{or } x^2y^2 - a^2xy = \frac{e - a^5}{5a};$$

and, by solving this quadratic, we obtain

$$xy = \frac{a^2}{2} \pm \sqrt{\frac{a^5 + 4e}{20a}};$$

$\therefore$ calling this value of xy, m, we have, from the equation,

$$x + y = a,$$

$$\frac{m}{y} + y = a, \text{ or } m + y^2 = ay, \therefore\ y^2 - ay = -m;$$

$$\therefore\ y = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - m} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a^2}{2} \mp \sqrt{\frac{a^5 + 4e}{20a}}}$$

$$= \frac{a}{2} \pm \sqrt{-\frac{a^2}{4} \mp \sqrt{\frac{a^5 + 4e}{20a}}};$$

$$\text{and } x = a - y = \frac{a}{2} \mp \sqrt{-\frac{a^2}{4} \mp \sqrt{\frac{a^5 + 4e}{20a}}}.$$

Or, thus:

Putting $x = s + z$, and $y = s - z$, as in the preceding examples,

$$x^5 = s^5 + 5s^4z + 10s^3z^2 + 10s^2z^3 + 5sz^4 + z^5,$$

$$y^5 = s^5 - 5s^4z + 10s^3z^2 - 10s^2z^3 + 5sz^4 - z^5;$$

$\therefore$ by addition, $x^5 + y^5 = 2s^5 + 20s^3z^2 + 10sz^4 \qquad = e,$

$$\text{or } z^4 + 2s^2z^2 + \frac{s^4}{5} = \frac{e}{10s};$$

and by transposing, and completing the square,

$$z^4 + 2s^2z^2 + s^4 = \frac{e}{10s} - \frac{s^4}{5} + s^4 = \frac{8s^5 + e}{10s};$$

$\therefore$ extracting the root, &c.

$$z^2 = -s^2 \pm \sqrt{\frac{8s^5 + e}{10s}}, \therefore z = \pm\sqrt{-s^2 \pm \sqrt{\frac{8s^5 + e}{10s}}};$$

whence, by restoring the value of s,

$$x = s + z = \frac{a}{2} \pm \sqrt{-\frac{a^2}{4} \pm \sqrt{\frac{\frac{a^5}{4} + e}{5a}}}$$

$$= \frac{a}{2} \pm \sqrt{-\frac{a^2}{4} \pm \sqrt{\frac{a^5 + 4e}{20a}}};$$

$$\text{and } \therefore y = a - x = \frac{a}{2} \mp \sqrt{-\frac{a^2}{4} \pm \sqrt{\frac{a^5 + 4e}{20a}}},$$

the same as before.*

5. Given $\left\{ \begin{matrix} x + y = s \\ xy = p \end{matrix} \right\}$, to find the values of $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, &c.

By squaring the first equation,

$$x^2 + 2xy + y^2 = s^2;$$

and subtracting twice the second, $\quad 2xy \quad = 2p;$

there results . . . . $\quad x^2 + y^2 = s^2 - 2p.$

$$\text{Again, } (x + y)(x^2 + y^2) = s^3 - ps$$

$$xy(x + y) = ps$$

$$\therefore x^3 + y^3 = s^3 - 3ps$$

* If we had given $x + y$, and $x^6 + y^6$, to find x and y, the question would be impossible in quadratics, since, as it is easy to perceive, the operation would lead to a cubic equation; we cannot, therefore, extend the above examples any further.

Also,
$$\begin{aligned} (x+y)(x^3+y^3) &= s^4 - 3ps^2 \\ xy(x^2+y^2) &= \quad\quad ps^2 - 2p^2 \\ \hline \therefore\ x^4+y^4 &= s^4 - 4ps^2 + 2p^2 \end{aligned}$$

In like manner,
$$\begin{aligned} (x+y)(x^4+y^4) &= s^5 - 4ps^3 + 2p^2s \\ xy(x^3+y^3) &= \quad\quad ps^3 - 3p^2s \\ \hline \therefore\ x^5+y^5 &= s^5 - 5ps^3 + 5p^2s \end{aligned}$$

By continuing this simple process, formulas may be deduced to any extent. These formulas, it may be remarked, would have enabled us to arrive at simpler solutions to the four preceding questions than those already given. Thus, taking the fourth question, we have by the formula last deduced,

$$a^5 - 5a^3p + 5ap^2 = e$$
$$\therefore\ 5ap^2 - 5a^3p = e - a^5$$

and, completing the square and extracting the root,

$$p = \tfrac{1}{2}a^2 \pm \tfrac{1}{2}\sqrt{\left\{\frac{a^5+4e}{5a}\right\}} = xy.$$

Now $x - y = \sqrt{a^2 - 4p}$, and half this added to $\frac{1}{2}a$ gives x, and subtracted from it gives y: hence

$$x = \tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{\left\{-a^2 \pm 2\sqrt{\frac{a^5+4e}{5a}}\right\}}.$$

Particular Examples.

1. Given the sum of two numbers equal to 24, and the sum of their squares equal to 306. To find the numbers.

Ans. 9 and 15.

2. The sum of two numbers is 27, and the sum of their cubes 4941. Required the numbers.

Ans. 13 and 14.

3. The sum of two numbers is 11, and the sum of their fourth powers 2657. What are the numbers?

Ans. 4 and 7.

4. The sum of two numbers is 10, and the sum of their fifth powers 17050. What are the numbers?

Ans. 3 and 7.

5. The sum of two numbers is 47, and their product 546. Required the sum of their squares.

Ans. 1117.

6. The sum of two numbers is 20, and their product 99. Required the sum of their cubes.

Ans. 2060.

7. The sum of two numbers is 19, and their product 78. What is the sum of their fourth powers?

Ans. 29857.

(97.) *When both Equations have a Quadratic Form.*

In this case, which includes every possible form, no general method of procedure can be pointed out; and the solution, in many cases, must be left to the ingenuity of the learner: many equations, however, that come under this case are irresolvable by *quadratics only*, and require equations of the higher degrees, as has been before observed.* When however the proposed quadratics are both *homogeneous*, that is, when every unknown term is of two dimensions, the solution may always be effected by adopting the artifice of substituting for one of the unknowns an unknown multiple of the other; because we shall thus introduce the square of this other into every term, and may therefore eliminate it from the equations. The result of this elimination will be a single quadratic, in which the unknown

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* The most general form under which quadratics containing two unknown quantities can be expressed is the following, viz.

$$ax^2 + by^2 + cxy + dx + ey = \text{A},$$
$$a'x^2 + b'y^2 + c'xy + d'x + e'y = \text{B};$$

the solution of which is a branch of the general doctrine of ELIMINATION, a subject too abstruse to be treated on fully in an elementary work like the present. The elimination of equations of the first degree has been already given, in Chap. II.; and, for those of the higher degrees, the reader is referred to the work of M. Bezout, before mentioned.

There is also a neat tract on this subject by M. Garnier, entitled "De l'Elimination entre deux Equations Algébriques d'un Degré quelconque."

is the assumed multiplier at first introduced, and the determination of which leads immediately to the solution. The most general form in which a pair of homogeneous quadratics can occur is the following, viz.

$$\left\{\begin{array}{l} ax^2 + bxy + cy^2 = d \\ a'x^2 + b'xy + c'y^2 = d' \end{array}\right\}^*.$$

To find the values of x and y in these equations, put $x = zy$, and they become

$$\left\{\begin{array}{l} az^2y^2 + bzy^2 + cy^2 = d \\ a'z^2y^2 + b'zy^2 + c'y^2 = d' \end{array}\right\};$$

from the first of which we get

$$y^2 = \frac{d}{az^2 + bz + c};$$

and from the second,

$$y^2 = \frac{d'}{a'z^2 + b'z + c'};$$

$$\text{whence } \frac{d}{az^2 + bz + c} = \frac{d'}{a'z^2 + b'z + c'};$$

or, clearing the equations of fractions,

$$a'dz^2 + b'dz + c'd = ad'z^2 + bd'z + cd':$$

a quadratic from which the values of z may be found, and, consequently, those of y and x may then be determined.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 4x^2 - 2xy = 12 \\ 2y^2 + 3xy = 8 \end{array}\right\}$, to find the values of x and y.

These equations being homogeneous, are resolvable by the above process; therefore, assuming $x = zy$, we have

$$4z^2y^2 - 2zy^2 = (4z - 2)\,zy^2 = 12,$$

$$\text{and } 2y^2 + 3zy^2 = (2 + 3z)\,y^2 = 8;$$

* This general form evidently includes a great variety of equations, since it comprehends all those in which any of the coefficients a, b, c, a', b', c', are $= 0$; that is, in which any of the terms are absent.

$\therefore$ from the first equation,

$$y^2 = \frac{12}{(4z - 2)z};$$

and from the second,

$$y^2 = \frac{8}{2 + 3z};$$

$$\therefore \frac{12}{(4z - 2)z} = \frac{8}{2 + 3z},$$

$$\text{or } 24 + 36z = 32z^2 - 16z;$$

and by transposition,

$$32z^2 - 52z = 24, \text{ or } z^2 - \tfrac{13}{8}z = \tfrac{3}{4};$$

$\therefore$ completing the square, and extracting the root,

$$z - \tfrac{13}{16} = \pm \tfrac{19}{16}, \therefore z = 2, \text{ or } -\tfrac{3}{8};$$

$$\text{and } y^2 = \frac{8}{2 + 3z} = 1, \text{ or } \tfrac{64}{7}.$$

$$\therefore y = \pm 1, \text{ or } \pm \sqrt{\tfrac{64}{7}};$$

$$\text{and } x = zy = \pm 2, \text{ or } \mp \tfrac{3}{8} . \sqrt{\tfrac{64}{7}}.$$

2. Given $\left\{ \begin{matrix} 6x^2 + 2y^2 = 5xy + 12 \\ 2xy + 3x^2 = 3y^2 - 3 \end{matrix} \right\}$, to find the values of x and y.

These equations being homogeneous, substitute, as before, zy for x, and we have

$$6z^2y^2 - 5zy^2 + 2y^2 = \{6z^2 - 5z + 2\}y^2 = 12;$$

$$\text{and } 2zy^2 - 3y^2 + 3z^2y^2 = \{2z - 3 + 3z^2\}y^2 = -3;$$

from the first of these equations,

$$y^2 = \frac{12}{6z^2 - 5z + 2};$$

and from the second,

$$y^2 = \frac{-3}{2z - 3 + 3z^2};$$

$$\therefore \frac{12}{6z^2 - 5z + 2} = \frac{-3}{2z - 3 + 3z^2},$$

$$\text{or } 24z - 36 + 36z^2 = -18z^2 + 15z - 6; \therefore 54z^2 + 9z = 30$$

$$\therefore 6z^2 + z = \tfrac{10}{3};$$

and completing the square (Art. 90),

$$144z^2 + 24z + 1 = 81;$$

and extracting,

$$12z + 1 = \pm 9;$$

$$\therefore\ z = \tfrac{2}{3},\ \text{or}\ -\tfrac{5}{6}:$$

$$\text{whence } y^2 = 9,\ \text{or}\ \tfrac{36}{31};\ \therefore\ y = \pm 3,\ \text{or}\ \pm\frac{6}{\sqrt{31}},$$

$$\text{and}\ \therefore\ x = \pm 2,\ \text{or}\ \mp\frac{5}{\sqrt{31}}.$$

3. Given $\left\{\begin{array}{l} x^2 + xy = 12 \\ xy - 2y^2 = 1 \end{array}\right\}$, to find the values of x and y.

$$\text{Ans.}\left\{\begin{array}{l} x = \pm 3,\ \text{or}\ \pm\dfrac{8}{\sqrt{6}}. \\ y = \pm 1,\ \text{or}\ \pm\dfrac{1}{\sqrt{6}}. \end{array}\right.$$

4. Given $\left\{\begin{array}{l} 3x^2 + xy = 68 \\ 4y^2 + 3xy = 160 \end{array}\right\}$, to find the values of x and y.

$$\text{Ans.}\left\{\begin{array}{l} x = \pm 4,\ \text{or}\ \mp\dfrac{17\sqrt{768}}{72}. \\ y = \pm 5,\ \text{or}\ \pm\dfrac{\sqrt{768}}{3}. \end{array}\right.$$

5. Given $\left\{\begin{array}{l} 2x^2 - 3xy + y^2 = 4 \\ 2xy - 3y^2 - x^2 = -9 \end{array}\right\}$, to find the values of x and y.

$$\text{Ans.}\left\{\begin{array}{l} x = \pm 3,\ \text{or}\ \mp\dfrac{1}{\sqrt{18}} \\ y = \pm 2,\ \text{or}\ \pm\dfrac{7}{\sqrt{18}}. \end{array}\right.$$

(98.) MISCELLANEOUS EXAMPLES.

To which the preceding Methods do not immediately apply.

1. Given $\left\{\begin{array}{l} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{array}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$x^2 + y^2 + x + y = 18;$$

and from the second, by multiplication,

$$2xy = 12;$$

$\therefore$ by addition, $x^2 + 2xy + y^2 + x + y = 30$;

and substituting in this equation the value of x $(= \frac{6}{y})$, as obtained from the second equation, it becomes

$$\left(\frac{6}{y} + y\right)^2 + \left(\frac{6}{y} + y\right) = 30,$$

or putting $\frac{6}{y} + y = z$, it is $z^2 + z = 30$;

$\therefore$ completing the square,

$$z^2 + z + \tfrac{1}{4} = \tfrac{121}{4};$$

and extracting the root,

$$z + \tfrac{1}{2} = \pm \tfrac{11}{2};$$

$$\therefore\ z, \text{ or } \frac{6}{y} + y, = 5, \text{ or } -6;$$

$$\text{whence } 6 + y^2 = 5y, \text{ or } -6y;$$

$$\therefore \begin{cases} y^2 - 5y = -6, \\ \text{or } y^2 + 6y = -6; \end{cases}$$

and completing the first square (Art. 90),

$$4y^2 - 20y + 25 = 1;$$

and extracting

$$2y - 5 = \pm 1;$$

$$\therefore\ y = 3, \text{ or } 2;$$

also, completing the second square,

$$y^2 + 6y + 9 = 3;$$

and extracting

$$y + 3 = \pm\sqrt{3};$$

$$\therefore\ y = -3 \pm \sqrt{3};$$

whence $x = \frac{6}{y} = 2$, or 3; or $-3 \mp \sqrt{3}$.

2. Given $\left\{\begin{matrix} x^2y - y = 21 \\ x^2y - xy = 6 \end{matrix}\right\}$, to find the values of x and y.

From the first equation,

$$y = \frac{21}{x^2 - 1};$$

and from the second,

$$y = \frac{6}{x^2 - x};$$

$$\therefore \frac{21}{x^3 - 1} = \frac{6}{x^2 - x}.$$

or dividing by $\frac{3}{x-1}$,

$$\frac{7}{x^2 + x + 1} = \frac{2}{x}, \therefore 7x = 2x^2 + 2x + 2,$$

and by transposition,

$$2x^2 - 5x = -2;$$

$\therefore$ completing the square (Art. 90),

$$16x^2 - 40x + 25 = 9;$$

and extracting the root,

$$4x - 5 = \pm 3;$$

$$\therefore x = 2, \text{ or } \tfrac{1}{2}:$$

$$\text{and } y = \frac{6}{x^2 - x} = 3, \text{ or } -24.$$

3. Given $\left\{\begin{matrix} x^2 + 3x + y = 73 - 2xy \\ y^2 + 3y + x = 44 \end{matrix}\right\}$, to find the values of x and y.

By transposition, the first equation becomes

$$x^2 + 2xy + 3x + y = 73,$$

to which, if the second equation be added, there results

$$x^2 + 2xy + y^2 + 4x + 4y = (x + y)^2 + 4(x + y) = 117;$$

and completing the square,

$$(x + y)^2 + 4(x + y) + 4 = 121;$$

$\therefore$ extracting the root,

$$(x + y) + 2 = \pm 11;$$

$$\therefore x + y = 9, \text{ or } -13;$$

$$\text{and } x = 9 - y, \text{ or } -13 - y:$$

and, by substituting these values of x in the second equation, we have

$$y^2 + 2y + 9 = 44,$$

$$\text{or } y^2 + 2y - 13 = 44;$$

$\therefore$ by transposing, and completing the square in the first equation,

$$y^2 + 2y + 1 = 36;$$

and extracting the root,

$$y + 1 = \pm 6;$$

$$\therefore y = 5, \text{ or } -7:$$

also by transposing, and completing the square in the second equation,

$$y^2 + 2y + 1 = 58;$$

and extracting the root,

$$y + 1 = \pm\sqrt{58};$$

$$\therefore y = -1 \pm \sqrt{58}:$$

hence the values of y are, $y = 5$, or -7; or $-1 \pm \sqrt{58}$:
and those of x are $\therefore$ $x = 4$, or 16; or $-12 \mp \sqrt{58}$.

4. Given $\left\{\begin{array}{l} x^2 - y^2 - (x+y) = 8 \\ (x-y)^2 \cdot (x+y) = 32 \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 4, we have

$$4\{x^2 - y^2 - (x+y)\} = (x-y)^2 \cdot (x+y);$$

and, dividing this by $x + y$, there results

$$4(x - y - 1) = (x - y)^2;$$

and by transposition,

$$(x-y)^2 - 4(x-y) = -4;$$

$\therefore$ completing the square,

$$(x-y)^2 - 4(x-y) + 4 = 0;$$

and extracting the root,

$$(x-y) - 2 = 0;$$

$$\therefore x - y = 2:$$

and this value of $x - y$ substituted in the second equation, gives

$$4(x+y) = 32, \therefore x + y = 8;$$

and by addition, $\left\{\begin{array}{l} x - y = 2 \\ x + y = 8 \end{array}\right\}$ also by subtraction $\left\{\begin{array}{l} x + y = 8, \\ x - y = 2, \end{array}\right.$

we get $2x = 10$; we get $2y = 6$;

whence $x = 5$; and $y = 3$.

5. Given $\left\{\begin{array}{l} \dfrac{x^2}{y}+\dfrac{y^2}{x}= \ a \\ x+y \ =2b \end{array}\right\}$, to find the values of x and y.

Assume $x = z + v$,

and $y = z - v$;

$$\therefore x + y = 2z = 2b;$$

$$\therefore z = b \ \therefore x = b + v, \ y = b - v.$$

Now from the first equation,

$$x^3 + y^3 = axy \ \ldots \ (1);$$

$$\text{but } x^3 = (b + v)^3 = b^3 + 3b^2v + 3bv^2 + v^3,$$

$$y^3 = (b - v)^3 = b^3 - 3b^2v + 3bv^2 - v^3;$$

$$\therefore x^3 + y^3 = 2b^3 + 6bv^2 \ \ldots \ (2).$$

Again,

$$axy = a(b + v)(b - v) = ab^2 - av^2 \ \ldots \ (3);$$

hence, substituting (2) and (3) in (1), we have

$$2b^3 + 6bv^2 = ab^2 - av^2;$$

$$\therefore (a + 6b)v^2 = ab^2 - 2b^3;$$

$$\therefore v^2 = \frac{b^2(a - 2b)}{a + 6b};$$

$$\therefore v = b\sqrt{\frac{a - 2b}{a + 6b}};$$

$$\therefore x = b + b\sqrt{\frac{a - 2b}{a + 6b}};$$

$$y = b - b\sqrt{\frac{a - 2b}{a + 6b}}.$$

6. Given $\left\{\begin{array}{l} x^2 + x = \dfrac{12}{y} \\ x^2y + y = 18 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2, \text{ or } \frac{1}{2}. \\ y = 2, \text{ or } 16. \end{array}\right.$

7. Given $\left\{\begin{array}{l} x^2 + 4y^2 = 256 - 4xy \\ 4y^2 - x^2 = 64 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \pm 6, \\ y = \pm 5. \end{array}\right.$

8. Given $\left\{\begin{array}{l}(x^2+y^2)\cdot(x-y)=51 \\ x^2+y^2+x \quad\quad =20+y\end{array}\right\}$, to find the values of x and y.

$$\text{Ans.}\left\{\begin{array}{l} x=4, \text{ or } -1;\text{ or } \dfrac{17\pm\sqrt{-283}}{2}, \\ y=1, \text{ or } -4;\text{ or } \dfrac{-17\pm\sqrt{-283}}{2}. \end{array}\right.$$

(99.) QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

QUESTION I.

It is required to find three numbers, such, that the difference of the first and second shall exceed the difference of the second and third by 6; and that their sum may be 33, and the sum of their squares 467.

Let x be the second number, and y the difference of the first and second;

then the first number will be $x-y$,

and the third, by the question, $x+y+6$;

$\therefore$ their sum $=3x+6=33$, $\therefore x=9$:

also $x^2+(x-y)^2+(x+y+6)^2=467$, $\therefore (x-y)^2+(x+y+6)^2$
$$=386;$$

$$\text{that is, } 2x^2+12x+12y+2y^2+36=386,$$

or substituting for x its value $=9$,

$$306+12y+2y^2=386;$$
$$\therefore\ y^2+6y=40;$$

and completing the square,

$$y^2+6y+9=49;$$

$\therefore$ extracting the root,

$$y+3=\pm 7,$$
$$\text{and } y=4, \text{ or } -10:$$

hence the three numbers are 5, 9, and 19; or rather 19, 9, and 5.

QUESTION II.

It is required to find three numbers in geometrical progression, such, that their sum shall be 14, and the sum of their squares 84.

Let $\frac{x}{y}$, x, and xy, be the three numbers;

then, by the question,

$$\frac{x}{y} + x + xy = 14,$$

$$\text{and } \frac{x^2}{y^2} + x^2 + x^2y^2 = 84;$$

∴ from the first equation,

$$\frac{x}{y} + xy = 14 - x;$$

or squaring each side,

$$\frac{x^2}{y^2} + 2x^2 + x^2y^2 = (14)^2 - 28x + x^2.$$

$$\therefore \frac{x^2}{y^2} + x^2 + x^2y^2 = (14)^2 - 28x;$$

and ∴ from the second equation, we have

$$84 = (14)^2 - 28x;$$

$$\therefore 6 = 14 - 2x, \text{ and } \therefore x = \frac{14-6}{2} = 4;$$

and substituting this value of x in the first equation,

$$\frac{4}{y} + 4 + 4y = 14;$$

$$\therefore 4y^2 - 10y = -4,$$

$$\text{or } y^2 - \tfrac{10}{4}y = -1;$$

and completing the square,

$$y^2 - \tfrac{10}{4}y + \tfrac{25}{16} = \tfrac{9}{16};$$

∴ extracting the root,

$$y - \tfrac{5}{4} = \pm \tfrac{3}{4};$$

$$\text{whence } y = 2, \text{ or } \tfrac{1}{2}:$$

∴ the three numbers are 2, 4, and 8.

ANOTHER SOLUTION.

Let x and y denote the two extremes, then $\sqrt{xy}$ is the mean, and by the question,

$$x + \sqrt{xy} + y = 14,$$

$$\text{and } x^2 + xy + y^2 = 84.$$

Dividing this equation by the former,

$$x - \sqrt{xy} + y = 6;$$

hence, by addition to the first,

$$x + y = 10;$$

and by subtraction,

$$\sqrt{xy} = 4, \text{ or } xy = 16:$$

consequently,

$$(x + y)^2 - 4xy = 100 - 64 = 36,$$

$$\therefore\ x - y = 6$$

$$x + y = 10$$

$$\therefore\ x = 8,\ y = 2$$

hence the numbers are 2, 4, and 8.

QUESTION III.

The sum of four numbers in arithmetical progression is 34, and the sum of their squares 334. What are the numbers?

Let the two means be $x + y$, and $x - y$;
then the extremes will be $x + 3y$, and $x - 3y$;
and their sum $= 4x = 34$, $\therefore\ x = \frac{17}{2}$:
also the sum of their squares $= 4x^2 + 20y^2 = 334$;
$\therefore$ substituting in this equation the value of x found above, we have

$$289 + 20y^2 = 334;$$

$$\therefore\ 20y^2 = 45;$$

$$\text{whence } y = \pm\sqrt{\tfrac{9}{4}} = \pm\tfrac{3}{2}:$$

$\therefore$ the four numbers are 13, 10, 7, and 4.

QUESTION IV.

The sum of three numbers in harmonical proportion is 13, and the product of their extremes is 18. What are the numbers?

Let the extremes be x and y;

then the mean will be $\frac{2xy}{x+y}$ (Art. 79, Ch. 3);

and their sum $= x + \frac{2xy}{x+y} + y = 13$;

also the product of the extremes $= xy = 18$;

$\therefore$ by substitution,

$$x + \frac{36}{x+y} + y = 13;$$

and multiplying by $x + y$, and transposing,

$$(x+y)^2 - 13(x+y) = -36;$$

$\therefore$ completing the square (Art. 90),

$$4(x+y)^2 - 52(x+y) + 169 = 25;$$

and extracting the root,

$$2(x+y) - 13 = \pm 5;$$

$$\therefore x + y = 9, \text{ or } 4;$$

$$\begin{array}{lrcl}
\text{whence} & (x+y)^2 & = & 81, \text{ or } 16: \\
\text{and subtracting} & 4xy & = & 72, \\
\hline
\text{we have} & (x-y)^2 & = & 9, \text{ or } -56; \\
 & \therefore x - y & = & 3, \text{ or } \pm\sqrt{-56}; \\
\text{and adding} & x + y & = & 9 \\
\hline
 & 2x & = & 12; \ \therefore x = 6; \\
\text{also by subtracting} & 2y & = & 6; \ \therefore y = 3:
\end{array}$$

hence the three numbers are 6, 4, and 3.

Otherwise:

Let the extremes be $x + y$, and $x - y$;

then the mean will be $\frac{x^2 - y^2}{x}$;

and their sum $= 2x + \dfrac{x^2 - y^2}{x} = 13;$

also the product of the extremes $= x^2 - y^2 = 18;$

$\therefore$ by substitution,

$$2x + \frac{18}{x} = 13;$$

and multiplying by x, and transposing,

$$2x^2 - 13x = -18;$$

$\therefore$ completing the square (Art. 90),

$$16x^2 - 104x + 169 = 25;$$

and extracting the root,

$$4x - 13 = \pm 5;$$

$$\therefore x = \tfrac{9}{2}, \text{ or } 2:$$

and substituting the first of these values in the equation $x^2 - y^2 = 18$, we have

$$\tfrac{81}{4} - y^2 = 18, \therefore y^2 = \tfrac{9}{4}, \text{ and } y = \tfrac{3}{2}:$$

hence the numbers are 6, 4, and 3, as above.

QUESTION V.

It is required to find four numbers in arithmetical progression, such, that the product of the extremes shall be 45, and the product of the means 77.

Let x be the first term, and y the common difference, then the numbers will be

$$x,\ x + y,\ x + 2y,\ x + 3y;$$

and by the question,

$$x^2 + 3xy = 45,$$

$$x^2 + 3xy + 2y^2 = 77;$$

$\therefore$ by subtraction . . . . . . $2y^2 = 32,$

$$\therefore y = 4:$$

hence the first equation becomes

$$x^2 + 12x = 45;$$

which solved, gives $x = 3$: hence the four numbers are

3, 7, 11, and 15.

QUESTION VI.

It is required to find two numbers, such, that their sum, product, and the difference of their squares may be equal to each other.

Let x represent the greater number, and y the less,

then, by the question, $\begin{cases} x + y = xy \\ x + y = x^2 - y^2 \end{cases}$

Dividing each member of the second equation by $x + y$, we have

$$1 = x - y, \therefore y = x - 1.$$

Substituting this value of y in the first equation,

$$2x - 1 = x^2 - x,$$

$$\therefore x^2 - 3x = -1,$$

which solved, gives

$$x = \frac{3 + \sqrt{5}}{2}, \therefore y = \frac{1 + \sqrt{5}}{2}.$$

7. There are two numbers, whose sum, multiplied by the greater, gives 144, and whose difference, multiplied by the less, gives 14. What are the numbers? Ans. 9 and 7.

8. What number is that, which, being divided by the product of its two digits, the quotient is 2, and if 27 be added to the number, the digits will be inverted? Ans. 36.

9. A grocer sold 80 pounds of mace and 100 pounds of cloves for 65*l.*, and finds that he has sold 60 pounds more of cloves for 20*l.* than of mace for 10*l.* What was the price of a pound of each?
Ans. 1 lb. of mace is 10*s.*, and 1 lb. of cloves 5*s.*

10. It is required to find three numbers, whose sum is 38, such, that the difference of the first and second shall exceed the difference of the second and third by 7, and the sum of whose squares is 634.
Ans. 3, 15, and 20.

11. There are three numbers in geometrical progression, whose sum is 52, and the sum of the extremes is to the mean as 10 to 3. What are the numbers? Ans. 4, 12, and 36.

12. It is required to find two numbers, such, that their product

shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Ans. $\frac{5 + \sqrt{5}}{4}$, and $\frac{\sqrt{5}}{2}$.

13. The product of five numbers in arithmetical progression is 10395, and their sum is 35. What are the numbers?

Ans. 11, 9, 7, 5, and 3.

14. The sum of three numbers in geometrical progression is 13, and the product of the mean and sum of the extremes is 30. What are the numbers? Ans. 1, 3, and 9.

15. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. Required the numbers. Ans. 234 and 104.

16. There are three numbers, the difference of whose differences is 5; their sum is 20, and their product 130. What are the numbers? Ans. 2, 5, and 13.

17. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only four revolutions more than the hind-wheel in going the same distance. Required the circumference of each.

Ans. The circumference of the fore-wheel is 4 yards; and of the hind-wheel 5 yards.

ON IRRATIONAL QUANTITIES, OR SURDS.

(100.) An Irrational Quantity, or Surd, as it is sometimes called, is a quantity affected by a radical sign, or a fractional index, without which it cannot be accurately expressed; the quantity itself not being susceptible of the extraction which the index denotes.

Thus, $\sqrt{2}$ is a surd, because, as 2 is not a square, its square root cannot be accurately extracted; also, $\sqrt{6}$, $3^{\frac{1}{3}}$, $6^{\frac{1}{3}}$, &c. are surds, since none of them are susceptible of the requisite extraction; and therefore cannot be otherwise accurately expressed.

THEOREM 1. The square root of a quantity cannot be partly rational and partly a quadratic surd.

For if $\sqrt{a} = b + \sqrt{c}$, then, by squaring each side, we shall have $a = b^2 + 2b\sqrt{c} + c$, and $\therefore\ 2b\sqrt{c} = a - b^2 - c$; and, consequently, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$; that is, an irrational quantity equal to a rational quantity, which is impossible.

THEOREM 2. In any equation, consisting of rational quantities and quadratic surds, the rational quantities on each side are equal, as also the irrational quantities.

Let the equation be $a + \sqrt{b} = x + \sqrt{y}$, then, if a be not equal to x, let it be equal to $x \pm m$, and then $x \pm m + \sqrt{b} = x + \sqrt{y}$, $\therefore \pm m + \sqrt{b} = \sqrt{y}$; that is, the square root of a quantity is partly rational, and partly a quadratic surd, which is impossible (Theor. 1).

THEOREM 3. If $\sqrt{(a + \sqrt{b})} = x + y$, then will $\sqrt{(a - \sqrt{b})} = x - y$: x and y being supposed to be one or both quadratic surds.

For since $a + \sqrt{b} = x^2 + 2xy + y^2$, and since x and y are one or both quadratic surds, $x^2 + y^2$ must be rational, and $2xy$ irrational; $\therefore$ (Theor. 2), $a = x^2 + y^2$, and $\sqrt{b} = 2xy$, consequently, $a - \sqrt{b} = x^2 + y^2 - 2xy = (x - y)^2$, $\therefore \sqrt{(a - \sqrt{b})} = x - y$.

REDUCTION OF SURDS.

(101.) PROBLEM I. *To reduce a Rational Quantity to the Form of a Surd.*

Raise the quantity to the power denoted by the root of the surd proposed; then the corresponding root of this power, expressed by means of the radical sign, or a fractional index, will be the given quantity under the proposed form.

EXAMPLES.

1. Reduce 2 to the form of the square root.

   Here $2^2 = 4$, $\therefore \sqrt{4} = 2$ under the proposed form.

2. Reduce $3x^2$ to the form of the cube root.

   Here $(3x^2)^3 = 27x^6$, $\therefore 3x^2 = \sqrt[3]{27x^6}$.

3. Reduce a^2x^3 to the form of the fifth root.

4. Reduce $\dfrac{x^2}{y^3}$ to the form of the fourth root.

5. Reduce $\dfrac{\sqrt{a}}{x}$ to the form of the cube root.

6. Reduce $a^{\frac{1}{3}}x^{\frac{1}{4}}$ to the form of the square root.

Problem II. *To reduce Surds expressing different Roots to equivalent ones expressing the same root.*

Bring the indices to a common denominator; then raise each quantity to the power denoted by the numerator of its index, and the common denominator will denote the root of each.

EXAMPLES.

1. Reduce $\sqrt{2}$ and $\sqrt[3]{4}$ to surds expressing the same root.

Here the indices, brought to a common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$; $\therefore$ the proposed quantities are the same as $2^{\frac{3}{6}}$ and $4^{\frac{2}{6}}$; or $\sqrt[6]{8}$ and $\sqrt[6]{16}$.

2. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$ to surds expressing the same root.

Here the indices brought to a common denominator, are $\frac{3}{6}$ and $\frac{4}{6}$; $\therefore$ the proposed quantities are equivalent to $a^{\frac{3}{6}}$ and $a^{\frac{4}{6}}$; or to $\sqrt[6]{a^3}$ and $\sqrt[6]{a^4}$.

3. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to surds expressing the same root.

4. Reduce $2\sqrt[3]{3}$ and $3\sqrt{2}$ to surds expressing the same root.

5. Reduce $6^{\frac{2}{3}}$ and $5^{\frac{3}{4}}$ to surds expressing the same root.

6. Reduce $x^{\frac{3}{2}}$ and $y^{\frac{1}{5}}$ to surds expressing the same root.

Problem III. *To reduce Surds to their most Simple Forms.*

Surds, which admit of simplification, may always be divided into two factors, one of which will contain a perfect power corresponding to the surd root:

Hence, to simplify such surds, extract the root of that factor which is the perfect power, and multiply this root by the other factor, with the proper radical sign prefixed.

EXAMPLES.

1. Reduce $\sqrt{a^2b}$ to its most simple form.

Here, since a^2 is a perfect square, $\sqrt{a^2b} = a\sqrt{b}$.

2. Reduce $\sqrt[3]{135}$ to its most simple form.

$\sqrt[3]{135} = \sqrt[3]{27 \times 5} = 3\sqrt[3]{5}$, the answer.

3. Reduce $5\sqrt{54}$ to its most simple form.

$5\sqrt{54} = 5\sqrt{9 \times 6} = 5 \times 3\sqrt{6} = 15\sqrt{6}$, the form required.

4. Reduce $3\sqrt[3]{108}$ to its most simple form.

5. Reduce $\sqrt[3]{ax^3 + bx^6}$ to its most simple form.

6. Reduce $\sqrt[3]{5(a^3 + a^4b)}$ to its most simple form.

(102.) If the surd be in the form of a fraction, it may be simplified by multiplying both numerator and denominator by some quantity that will make the denominator of the requisite power; as in these

EXAMPLES.

1. Reduce $\sqrt{\frac{3}{8}}$ to its most simple form.

Here $\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16} \times 6} = \frac{1}{4}\sqrt{6}$.

2. Reduce $\frac{1}{2}\sqrt{\frac{3}{7}}$ to its most simple form.

$\frac{1}{2}\sqrt{\frac{3}{7}} = \frac{1}{2}\sqrt{\frac{21}{49}} = \frac{1}{2}\sqrt{\frac{1}{49} \times 21} = \frac{1}{14}\sqrt{21}$.

3. Reduce $\frac{a}{b}\sqrt{\frac{c^2}{d}}$ to its most simple form.

4. Reduce $5\sqrt[3]{\frac{2}{3}}$ to its most simple form.

5. Reduce $\frac{1}{2}\sqrt[3]{\frac{5}{9}}$ to its most simple form.

6. Reduce $\sqrt{\frac{ab^2}{4(a + x)}}$ to its most simple form.

ADDITION AND SUBTRACTION OF SURDS.

(103.) Reduce the surds to their most simple forms; then, if the surd part be the same in both, add or subtract the rational parts,* and annex the common surd part to the result: but if the surd parts be different, then the addition or subtraction can only be represented by the proper signs, $+$ or $-$.

EXAMPLES.

1. What is the sum of $\sqrt{18}$ and $\sqrt{8}$?

$$\left.\begin{array}{r} \text{Here } \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2} \\ \text{and } \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2} \end{array}\right\}, \therefore 5\sqrt{2} = \text{sum}.$$

2. What is the difference between $\sqrt{108ax^2}$ and $\sqrt{48ax^2}$?

$$\left.\begin{array}{r} \sqrt{108ax^2} = \sqrt{36x^2 \times 3a} = 6x\sqrt{3a} \\ \text{and } \sqrt{48ax^2} = \sqrt{16x^2 \times 3a} = 4x\sqrt{3a} \end{array}\right\}, \therefore 2x\sqrt{3a} = \text{difference}.$$

3. What is the sum of $3\sqrt[3]{32}$ and $2\sqrt[3]{54}$?

$$\left.\begin{array}{r} 3\sqrt[3]{32} = 3\sqrt[3]{8 \times 4} = 6\sqrt[3]{4} \\ \text{and } 2\sqrt[3]{54} = 2\sqrt[3]{27 \times 2} = 6\sqrt[3]{2} \end{array}\right\}, \therefore 6\sqrt[3]{4} + 6\sqrt[3]{2} = \text{sum}.$$

4. What is the difference between $\sqrt[3]{192}$ and $\sqrt[3]{24}$?

Ans. $2\sqrt[3]{3}$.

5. What is the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$?

Ans. $\frac{4}{5}\sqrt{10}$.

6. What is the difference between $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$?

Ans. $\frac{1}{18}\sqrt{6}$.

7. What is the sum of $\sqrt{24}$, $2\sqrt{72}$, and $a\sqrt{bx^2}$?

Ans. $2\sqrt{6} + 12\sqrt{2} + ax\sqrt{b}$.

8. Required the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

Ans. $8\sqrt[3]{4}$.

9. Required the difference between $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

Ans. $\frac{2}{5}\sqrt{10}$.

10. Required the difference between $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{1}{6}}$.

Ans. $\frac{11}{60}\sqrt{6}$.

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* The rational part is called the *coefficient* of the surd.

11. What is the difference between $5\sqrt{20}$ and $3\sqrt{45}$.

Ans. $\sqrt{5}$.

12. What is the sum of $\sqrt{27}$, $\sqrt{48}$, $4\sqrt{147}$, and $3\sqrt{75}$.

Ans. $50\sqrt{3}$.

MULTIPLICATION AND DIVISION OF SURDS.

(104.) Reduce the surds to equivalent ones expressing the same root (Prob. 2), and then multiply or divide as required.

EXAMPLES.

1. Multiply $\sqrt{8}$ by $\sqrt[3]{16}$.

Here $8^{\frac{1}{2}} \times (16)^{\frac{1}{3}} = 8^{\frac{3}{6}} \times (16)^{\frac{2}{6}} = \sqrt[6]{8^3 \times (16)^2} = \sqrt[6]{512 \times 256}$

$= \sqrt[6]{8(2)^6 \times 4(2)^6} = \sqrt[6]{32(4)^6} = 4\sqrt[6]{32} =$ product.

2. Divide $\sqrt{12}$ by $\sqrt[3]{24}$.

Here $\dfrac{(12)^{\frac{1}{2}}}{(24)^{\frac{1}{3}}} = \dfrac{(12)^{\frac{3}{6}}}{(24)^{\frac{2}{6}}} = \sqrt[6]{\dfrac{(12)^3}{(24)^2}} = \sqrt[6]{\dfrac{27(2)^6}{9(2)^6}} = 6\sqrt[6]{3} =$ quotient.

3. Multiply $2\sqrt[3]{\frac{2}{3}}$ by $3\sqrt[3]{\frac{5}{6}}$.

Ans. $2\sqrt[3]{15}$.

4. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$.

Ans. $\frac{4}{3}\sqrt[6]{\dfrac{a^2}{xy^3}}$.

5. Multiply $4\sqrt{3}$ by $3\sqrt[3]{4}$.

Ans. $12\sqrt[6]{432}$.

6. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$.

Ans. $2\sqrt{2}$.

7. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$.

Ans. $15\sqrt[6]{a^5}$.

8. Multiply $2\sqrt{27}$ by $\sqrt{3}$.

Ans. 18.

9. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{2}{3}\sqrt{2}$.

Ans. $\frac{3}{8}\sqrt{10}$.

To extract the Square Root of a Binomial Surd.

(105.) A binomial surd is that in which one of the terms, at least, is irrational; as $a \pm \sqrt{b}$, or $\sqrt{a} \pm \sqrt{b}$.*

(106.) In order to extract the square root of $a+\sqrt{b}$, put $\sqrt{(a+\sqrt{b})} = x + y$; and it follows that $\sqrt{(a-\sqrt{b})} = x - y$ (Art. 100, Theo. 3).

Let each of these equations be squared, and we have

$$\begin{array}{rl} a + \sqrt{b} &= x^2 + 2xy + y^2 \\ a - \sqrt{b} &= x^2 - 2xy + y^2; \\ \hline \therefore \text{ by addition} \,.\,.\, 2a &= 2x^2 + 2y^2, \text{ or } a = x^2 + y^2. \end{array}$$

Let the same two equations be now multiplied together, and there results

$$\sqrt{(a+\sqrt{b})} \times \sqrt{(a-\sqrt{b})} = x^2 - y^2, \text{ or } \sqrt{(a^2-b)} = x^2 - y^2;$$

hence, both the sum and difference of x^2 and y^2 being given, we have, by addition and subtraction,

$$x^2 = \frac{a + \sqrt{(a^2-b)}}{2}, \text{ and } y^2 = \frac{a - \sqrt{(a^2-b)}}{2};$$

$$\therefore\ x = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}, \text{ and } y = \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}};$$

consequently,

$$\sqrt{(a+\sqrt{b})} = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} + \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}$$

$$\sqrt{(a-\sqrt{b})} = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} - \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}.$$

(107.) In order that the expressions within the brackets may be rational, it is evident that both a and $\sqrt{(a^2-b)}$ must be rational; in which case, each of the above values will consist either of two surds, or of a rational part and a surd.

The above formulæ will apply to any particular example, by substituting the particular values for a and b; observing, that if b be negative, the signs of b in the formulæ are to be changed.

* The term *binomial* is often confined solely to surds of the form $a+\sqrt{b}$, or $\sqrt{a}+\sqrt{b}$; and those of the form $a-\sqrt{b}$, or $\sqrt{a}-\sqrt{b}$, are called *residual* surds.

EXAMPLES.

1. What is the square root of $8 + \sqrt{39}$?

Here $a = 8$, and $b = 39$;

$$\therefore \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{8+\sqrt{(8^2-39)}}{2}\right\}} = \sqrt{\tfrac{13}{2}};$$

$$\text{and } \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{8-\sqrt{(8^2-39)}}{2}\right\}} = \sqrt{\tfrac{3}{2}};$$

$$\therefore \sqrt{(8+\sqrt{39})} = \sqrt{\tfrac{13}{2}} + \sqrt{\tfrac{3}{2}}.$$

2. What is the square root of $10 - \sqrt{96}$?

Here $a = 10$, and $b = 96$;

$$\therefore \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{10+\sqrt{(10^2-96)}}{2}\right\}} = \sqrt{6};$$

$$\text{and } \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{10-\sqrt{(10^2-96)}}{2}\right\}} = 2;$$

$$\therefore \sqrt{(10-\sqrt{96})} = \sqrt{6} - 2.$$

3. What is the square root of $6 + \sqrt{20}$?

Ans. $1 + \sqrt{5}$.

4. What is the square root of $6 - 2\sqrt{5}$?

Ans. $\sqrt{5} - 1$.

5. What is the square root of $7 - 2\sqrt{10}$?

Ans. $\sqrt{5} - \sqrt{2}$.

6. What is the square root of $42 + 3\sqrt{174\frac{2}{9}}$?

Ans. $\sqrt{28} + \sqrt{14}$.

To find Multipliers which will make Binomial Surds rational.

(108.) Surds may often be rendered rational by being multiplied by some other quantity, which quantity, when the surd consists of but a simple term, is always easily found: if, for instance, the surd $\sqrt{a}$ is to be freed from its irrational form, it must evidently be multiplied by $\sqrt{a}$; for $\sqrt{a} \times \sqrt{a} = a$; and if $\sqrt[3]{a}$ be the form of the surd, then the multiplier must be $\sqrt[3]{a^2}$; because $\sqrt[3]{a} \times \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$: and, generally, the multiplier that will make $\sqrt[n]{a}$ rational, is $\sqrt[n]{a^{n-1}}$; because $\sqrt[n]{a} \times \sqrt[n]{a^{n-1}} = \sqrt[n]{a^n} = a$. The more usual *bino-*

mial forms may too be readily rationalized; such forms consist of either the sum or difference of two *square roots*, or else of the sum or difference of two *cube roots*. In the former case the multiplier will be suggested from the property, that the product of the sum and difference of two quantities is the difference of their squares. In the latter case the multiplier will be a *trinomial surd*, consisting of the squares of the two given terms, and of their product with its sign changed; that is to say, the form $\sqrt[3]{a} \pm \sqrt[3]{b}$ will be rendered rational by the multiplier $\sqrt[3]{a^2} \mp \sqrt[3]{ab} + \sqrt[3]{b^2}$, since it is plain that the extreme terms of the product will be rational, and that the four intermediate terms destroy each other. But it is not so easy to discover, at once, the multiplier that will render *any binomial* surd rational; the method of proceeding, however, in this case, is derived from the following investigation:

$$\text{By division}\begin{cases}\dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \text{\&c.} \\[2ex] \dfrac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^2 + \text{\&c.} \\[2ex] \dfrac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \text{\&c.}\end{cases}$$

Here the first of these series will terminate at the nth term, whether n be even or odd; the second will terminate at the nth term, only when n is an even number; and the third, only when n is an odd number; for, in other cases, they will go on to infinity, as will appear by substituting different numbers successively for n.*

Now put $x^n = a$, $y^n = b$; then $x = \sqrt[n]{a}$, and $y = \sqrt[n]{b}$; and the above fractions become, respectively,

$$\frac{a - b}{\sqrt[n]{a} - \sqrt[n]{b}}, \quad \frac{a - b}{\sqrt[n]{a} + \sqrt[n]{b}}, \text{ and } \frac{a + b}{\sqrt[n]{a} + \sqrt[n]{b}};$$

and since $x = \sqrt[n]{a}$, $x^{n-1} = \sqrt[n]{a^{n-1}}$; $x^{n-2} = \sqrt[n]{a^{n-2}}$, &c.;

$$\text{also } y^2 = \sqrt[n]{b^2}; \ y^3 = \sqrt[n]{b^3}, \text{ \&c.};$$

and, substituting these values in the above quotients,

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* We cannot prove the truth of these properties otherwise than by in duction in this place; since the general demonstration of them requires the Binomial Theorem. See Barlow's Theory of Numbers, chapter 6.

we have
$$\left\{\begin{aligned}
\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}} &= \sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\sqrt[n]{a^{n-4}b^3}+\ \&c.\\
\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}} &= \sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ \&c.\\
\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}} &= \sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ \&c.
\end{aligned}\right.$$

Therefore, since the divisor multiplied by the quotient produces the dividend, it follows, that if a surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$, be multiplied by

$$\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\ \&c.$$

to n terms, the product will be $a-b$, a rational quantity; also, if a surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$, be multiplied by $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-$&c. to n terms, the product will be $a-b$, or $a+b$, according as n is even, or odd, each of which products is a rational quantity.

The Examples which follow, although worked by the general formulas here given, may nevertheless be readily solved by the two simple rules stated at the commencement of this article.

EXAMPLES.

1. It is required to find a multiplier that shall make $\sqrt[3]{7}-\sqrt[3]{5}$ rational.

Here $a=7$, $b=5$, and $n=3$, $\therefore$ the multiplier, $\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+$ &c. $=\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}$,

$$\begin{array}{ll}
 & \sqrt[3]{7}-\sqrt[3]{5} \\
\hline
\text{and by actual multiplication} \ldots\ldots & \sqrt[3]{343}+\sqrt[3]{245}+\sqrt[3]{175}, \\
 & \qquad -\sqrt[3]{245}-\sqrt[3]{175}-\sqrt[3]{125} \\
\hline
\text{there results} \ldots\ldots & \sqrt[3]{343} \quad * \quad * \quad -\sqrt{125}=7-5, \\
\hline
\end{array}$$

as there ought.

2. It is required to find a multiplier that will make $2+\sqrt[3]{3}$ rational.

Here $2+\sqrt[3]{3}=\sqrt[3]{8}+\sqrt[3]{3}$, $\therefore a=8$, $b=3$, and $n=3$, $\therefore$ the multiplier, $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+$ &c.

$$=\sqrt[3]{64}-\sqrt[3]{24}+\sqrt[3]{9}=4-\sqrt[3]{24}+\sqrt[3]{9}.$$

3. It is required to convert $\dfrac{3}{\sqrt[3]{5}-\sqrt[3]{2}}$ into a fraction that shall have a rational denominator.

$$\text{Ans. } \frac{3(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}{5-2}=\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}.$$

4. It is required to convert $\dfrac{a}{\sqrt{a}+\sqrt{b}}$ into a fraction that shall have a rational denominator.

$$\text{Ans. } \frac{a(\sqrt{a}-\sqrt{b})}{a-b}.$$

5. It is required to convert $\dfrac{a}{\sqrt[3]{x}+\sqrt[3]{y}}$ into a fraction that shall have a rational denominator.

$$\text{Ans. } \frac{a(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}{x+y}.$$

6. It is required to find a multiplier that will make $\sqrt[4]{3}+\sqrt[4]{4}$ rational.

$$\text{Ans. } \sqrt[4]{27}-\sqrt[4]{36}+\sqrt[4]{48}-\sqrt[4]{64}.$$

The answer here given to this last example has, like those above, been determined from the general formulas; but the proper multiplier may be more readily obtained, and in a preferable form, by the rule at the commencement of the article: thus the multiplier $\sqrt[4]{3}-\sqrt[4]{4}$ will bring the proposed form to $\sqrt{3}-\sqrt{4}$, and the multiplier $\sqrt{3}+\sqrt{4}$ reduces this last to $3-4$, so that the complete multiplier sought consists of the two factors $\sqrt[4]{3}-\sqrt[4]{4}$ and $\sqrt{3}+\sqrt{4}$, which produce the multiplier in the answer.

ON IMAGINARY QUANTITIES.

(109.) Imaginary quantities are those expressions which represent any even root of a negative quantity, as $\sqrt{-a}$, $\sqrt[4]{-a}$, &c. the values of such expressions being unassignable. These quanti-

ties differ from other surd expressions, inasmuch as the values of the latter, though inexpressible accurately, may still be approximated to; but imaginaries are not susceptible even of approximate values: notwithstanding this, however, they are of considerable use in various parts of the mathematics, and, when subjected to the ordinary rules of calculation, often lead to possible and valuable results.

(110.) With respect to the addition and subtraction of these quantities, the operations are the same as for quantities in general; but, as regards their multiplication and division, several particulars must be attended to that do not attach to other quantities; and which we shall here enter upon.

(111.) It is evident, in the first place, that $\sqrt{-a} \times \sqrt{-a}$ must be equal to $-a$; for the square root of any quantity multiplied by that square root, must produce the original quantity, and therefore no ambiguity can here arise with respect to the sign of a. It is also equally evident that $\sqrt{-a} \times \sqrt{-a}$ must be equal to $\sqrt{a^2}$; for if this be not the case, the rule for the signs in multiplication is not general: it therefore follows, that $-a$ must be equal to $\sqrt{a^2}$.

But it may be said that $\sqrt{a^2}$ is also $= a$, and that therefore it would follow that $a = -a$: this reasoning is, however, erroneous, for it is not true that $\sqrt{a^2}$ is *also* $= a$, since the symbol $\sqrt{}$ does not contain *both* the signs $+$ and $-$, but *either* $+$ or $-$, and, consequently, if it be shown to contain the one, it cannot at the same time *also* contain the other; in the present case, therefore, $\sqrt{}$ contains only the *minus* sign, and, consequently,

$$\sqrt{-a} \times \sqrt{-a} = -\sqrt{a^2} = -a.$$

(112.) Our being able to destroy the ambiguity of the symbol $\sqrt{}$ in the expression $\sqrt{a^2}$, arose solely from our previous knowledge of the manner in which a^2 was produced, viz. from the involution of $-a$; and that therefore the reverse operation being performed on a^2, must bring back the original quantity $-a$. If we had had no knowledge of the generation of a^2, whether it was produced from $(+a) \times (+a)$, or from $(-a) \times (-a)$; that is, whether a^2 represented $(+a)^2$, or $(-a)^2$; then, in the reverse operation, we could of course have had no knowledge of the precise quantity which ought to have been produced; that is, the symbol of extraction would

have been ambiguous, and the operation could only have been expressed by saying $\sqrt{a^2} = +a$, or $-a$.

In the same manner, if it be known that a^2 is produced from $(+a) \times (+a)$, then $\sqrt{a^2} = +\sqrt{a^2} = +a$.

(113.) Again, if we have two *unequal* imaginary quantities, $\sqrt{-a}$ and $\sqrt{-b}$, we know that their product, $\sqrt{-a} \times \sqrt{-b} = \sqrt{ab}$; but we do not immediately perceive whether this result is to be taken *positively*, or *negatively;* because here the quantity, whose root is to be extracted, was not generated from that root, but from two *unequal* factors; the proper sign may, nevertheless, be determined, for since

$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}$, and $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$, we have

$$\sqrt{-a} \times \sqrt{-b} = (\sqrt{a} \times \sqrt{-1}) \cdot (\sqrt{b} \times \sqrt{-1})$$
$$= \sqrt{ab} \times -1 = -\sqrt{ab}:$$

Hence it appears that the proper sign of $\sqrt{ab}$ is *minus:* and thus may any imaginary be represented by two factors, of which one is a real quantity, and the other the imaginary $\sqrt{-1}$; and therefore the expression, $\sqrt{-1}$, may be considered as a universal factor of every imaginary quantity, the other factor being a real quantity, either rational or irrational.

(114.) From what has been just said, and from the property that the multiplication of like signs always produces *plus*, it follows that,

The product of two imaginaries that have the same sign, is equal to *minus* the square root of their product, considering them as real quantities. That is,

$$(+\sqrt{-a})\,(+\sqrt{-a}) = -\sqrt{a^2} = -a;$$

as also

$$(-\sqrt{-a})\,(-\sqrt{-a}) = -\sqrt{a^2} = -a;$$

and

$$(+\sqrt{-a})\,(+\sqrt{-b}) = -\sqrt{ab};$$

as also

$$(-\sqrt{-a})\,(-\sqrt{-b}) = -\sqrt{ab}.$$

(115.) But if the two imaginaries have different signs, then their product will evidently be equal to *plus* the square root of their product, considering them as real. That is,

$$+\sqrt{-a})\,(-\sqrt{-b}) = +\sqrt{ab}.$$

EXAMPLES.

1. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

   Here $2\sqrt{-3} \times 3\sqrt{-2} = -6\sqrt{6}$.

2. Multiply $3+\sqrt{-2}$ by $2-\sqrt{-4}$.

$$\begin{array}{l} 3+\sqrt{-2} \\ 2-\sqrt{-4} \\ \hline 6+2\sqrt{-2} \\ \quad -3\sqrt{-4}+\sqrt{8} \\ \hline 6+2\sqrt{-2}-3\sqrt{-4}+\sqrt{8}. \end{array}$$

3. Multiply $4\sqrt{-5}$ by $3\sqrt{-1}$.

   Ans. $-12\sqrt{5}$.

4. Multiply $-5\sqrt{-2}$ by $-3\sqrt{-5}$.

   Ans. $-15\sqrt{10}$.

5. Multiply $4+\sqrt{-3}$ by $\sqrt{-5}$.

   Ans. $4\sqrt{-5}-\sqrt{15}$.

6. Required the cube of $a-b\sqrt{-1}$.

   Ans. $a^3+b^3\sqrt{-1}-3ab\,(b+a\sqrt{-1})$.

(116.) The quotient of two imaginaries having the same sign, is equal to *plus* the square root of their quotient, considering them as real quantities. That is,

$$\frac{+\sqrt{-a}}{+\sqrt{-b}} = \frac{+\sqrt{a}\times\sqrt{-1}}{+\sqrt{b}\times\sqrt{-1}} = +\sqrt{\frac{a}{b}};$$

as also

$$\frac{-\sqrt{-a}}{-\sqrt{-b}} = \frac{-\sqrt{a}\times\sqrt{-1}}{-\sqrt{b}\times\sqrt{-1}} = +\sqrt{\frac{a}{b}}.$$

(117.) But if the two imaginaries have different signs, it is evident that their quotient will be equal to *minus* the square root of their quotient, considering them as real quantities.

EXAMPLES.

1. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$.

$$\frac{6\sqrt{-3}}{2\sqrt{-4}} = 3\sqrt{\tfrac{3}{4}}.$$

2. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

Here the multiplier that will render $1-\sqrt{-1}$ rational is $1+\sqrt{-1}$ (Art. 108),

$$\therefore \frac{1+\sqrt{-1}}{1-\sqrt{-1}} = \frac{2\sqrt{-1}}{2} = \sqrt{-1}.$$

3. Divide $2\sqrt{-7}$ by $-3\sqrt{-5}$.

Ans. $-\tfrac{2}{3}\sqrt{\tfrac{7}{5}}$.

4. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$.

Ans. $+\dfrac{1}{6\sqrt{3}}$.

5. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

Ans. $1+\sqrt{-2}$.

6. Divide $3+2\sqrt{-1}$ by $3-2\sqrt{-1}$.

Ans. $\frac{1}{13}(5+12\sqrt{-1})$.

SCHOLIUM.

(118.) The arithmetic of imaginary, or impossible quantities, has been a subject of much disagreement among mathematicians; some affirming that the operations of these quantities are altogether absurd, and others, though admitting the validity of the operations, differing in their opinions of the results which they ought to produce. The most celebrated among the latter are Emerson and Euler, the for-

mer asserting the product of $\sqrt{-a}$ and $\sqrt{-b}$ to be $\sqrt{-ab}$,* and the latter making it $+\sqrt{ab}$†; and yet they both agree that $(\sqrt{-a}) \cdot (\sqrt{-a}) = -a$: hence it appears that they not only differ from each other, but even from themselves; for if, according to Emerson, $(\sqrt{-a}) \cdot (\sqrt{-b}) = \sqrt{-ab}$, whatever be the values of a and b, the equality must still subsist when $b = a$; that is, $(\sqrt{-a}) \cdot (\sqrt{-a}) = \sqrt{-a^2}$, an imaginary quantity; but the same product has been admitted to produce a real quantity, $-a$; therefore one of the conclusions must be absurd. The same inconsistency attaches to Euler's supposition, for, according to him, when $b = a$, we should have $+\sqrt{a^2} = -a$, which is also absurd.

(119.) The methods of operating on these quantities, as pointed out in the preceding articles, are free from the inconsistencies here noticed, and differ from the operations performed on quantities in general, only as it respects the restricted and definite sense in which the symbol $\sqrt{}$ is to be taken in their results, it sometimes meaning $+\sqrt{}$, and at other times $-\sqrt{}$, but never indifferently $\pm\sqrt{}$, as in the ordinary operations on real quantity.

(120.) We might also have shown that when a real and an imaginary quantity were to be multiplied together, or to be divided the one by the other, that similar processes to those already given were applicable; but, as the results would always be imaginary, no benefit would be derived from performing the actual operation: by way of elucidation, however, let it be required to find the product of $\sqrt[4]{a}$ and $\sqrt{-1}$. By the ordinary rule,

$$\sqrt[4]{a} \times \sqrt{-1} = \sqrt[4]{a} \times \sqrt[4]{(-1)^2} = \sqrt[4]{a} \times \sqrt[4]{1} = \sqrt[4]{a},$$

* "Thus, $\sqrt{-a} \times \sqrt{-b} = \sqrt{-ab}$, and $\sqrt{-a} \times -\sqrt{-b} = -\sqrt{-ab}$, &c. Also, $\sqrt{-a} \times \sqrt{-a} = -a$, and $\sqrt{-a} \times -\sqrt{-a} = +a$," &c.—EMERSON'S ALGEBRA, p. 67.

† "The product of $\sqrt{-3}$ by $\sqrt{-3}$ must be -3; the product of $\sqrt{-1}$ by $\sqrt{-1}$ is -1; and, in general, by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$, we obtain $-a$." "Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$ or 2 for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$."—EULER'S ALGEBRA.

but this result, in its unrestricted sense, is the same as $\pm\sqrt{}(\pm\sqrt{a})$, and, in order to arrive at its definite signification in the present case, we have

$$\sqrt[4]{a} \times \sqrt[4]{(-1)^2} = \sqrt{}(\sqrt{a} \times \sqrt{(-1)^2}) = \sqrt{}(\sqrt{a} \times -1)$$
$$= \sqrt{}(-\sqrt{a}).$$

(121.) It may be here further remarked, that imaginary quantities always occur in the analysis of a problem, when its conditions involve any absurdity or impossibility; as if it were proposed to divide the number 12 into two parts, such, that their product may be 40. If this question be solved by the ordinary rules, the two parts will be found to be $6+\sqrt{-4}$, and $6-\sqrt{-4}$, being both imaginary, or impossible in numbers. But, besides this, the use of imaginaries, as we have before said, is very extensive in some of the higher branches of analysis, and their application to a variety of highly interesting particulars has lately been shown by Benjamin Gompertz, Esq., F. R. S., &c. in his Tracts on "The Principles and Application of Imaginary Quantities."*

* Some modern writers disapprove of these expressions being called *quantities*; but, as they are susceptible of the same operations as quantities in general, and do often lead to results whose values are assignable, there appears to be no impropriety in the appellation. The term *imaginary*, however, does seem to require some modification; for, although the values of the quantities so called are *unassignable*, and even *inconceivable*, yet, speaking according to the common acceptation of the term, it seems saying too much to call them *imaginary*.

Lacroix, and those who say these expressions are not representatives of *quantities*, call them *imaginary expressions*, or *imaginary symbols*; but these appellations are still more objectionable, for as *expressions*, or *symbols*, they are certainly not *imaginary*, but *real*.

CHAPTER V.

ON THE BINOMIAL THEOREM; AND ITS APPLICATION TO THE EXPANSION OF SERIES, &c.

(122.) THE BINOMIAL THEOREM is a theorem discovered by Sir Isaac Newton,* whereby any power, or root, of a binomial may be obtained without the labour of performing the actual involution, or extraction. The power or root so found, is usually called the *expansion* of the binomial; but, before we proceed to investigate the form, or law of this expansion, it will be necessary to prove the truth of the following theorem:

THEOREM.

(123.) If the series $A + Bx + Cx^2 + Dx^3 +$ &c. be equal to the series $A' + B'x + C'x^2 + D'x^3 +$ &c., whatever be the value of x, then the coefficients of the *like* powers of x will be respectively equal the one to the other; that is,

$$A = A', \quad B = B', \quad C = C', \text{ \&c.}$$

For since the two series are equal, whatever be the value of x, they are equal when $x = 0$; but, in this case, the first series becomes simply A, and the second becomes A'; therefore, $A = A'$: and it also follows, that

$$Bx + Cx^2 + Dx^3 + \text{\&c.} = B'x + C'x^2 + D'x^3 + \text{\&c.};$$

* It appears to be not quite correct to ascribe the first discovery of this celebrated theorem to Sir Isaac Newton, as it was known and applied in the case of integral powers before his time. (See Dr. Hutton's History of Logarithms). Newton, however, was undoubtedly the first discoverer of the theorem under its present form, since none of his predecessors had ever shown its application in the cases of fractional, or negative exponents. It is remarkable, that, although this theorem was one of Newton's earliest discoveries, he has left no demonstration of it; and he is therefore supposed to have inferred its generality from an induction of particular cases. (See Biot's Life of Newton.)

whence, dividing by x, and supposing $x = 0$, we have $B = B'$; and, by proceeding in a similar manner, it may be shown that $C = C'$, $D = D'$, &c.

INVESTIGATION OF THE BINOMIAL THEOREM.

(124.) Let m be any positive number, either whole, or fractional; and let it be required to exhibit the expansion of $(a + x)^m$, or of its equal $a^m(1 + \frac{x}{a})^m$.

1. In the first place, since every power, or root of 1 is 1, the first term in the expansion of $(1 + \frac{x}{a})^m$ must be 1; and, consequently, the first term in the expansion of $a^m(1 + \frac{x}{a})^m$, or $(a + x)^m$ must be a^m.

2. Hence it follows, that the first term in the expansion of $\frac{1}{(a+x)^m}$, or of $(a + x)^{-m}$, must be $\frac{1}{a^m}$, or a^{-m}.

3. Therefore we may conclude, that whether m be whole or fractional, positive or negative, the first term in the expansion of $(a+x)^m$ must always be a^m.

(125.) Let now the exponent m, of the binomial $a+x$, be increased by 1; then the expansion of $(a + x)^{m+1}$ will be equal to each term in the expansion of $(a + x)^m$ multiplied by $a + x$; now the first term in the expansion of $(a + x)^m$ has been shown to be a^m, therefore $a^m \times (a + x) = a^{m+1} + a^m x =$ the first term, and the literal part of the second, in the expansion of $(a + x)^{m+1}$; but as the complete second term may, for aught we know to the contrary, have a coefficient, B, it may be represented by $Ba^m x$, where B will be $= 1$, should there be no coefficient. Let the exponent of the binomial be again increased by 1 then the expansion of $(a + x)^{m+2}$ will be equal to each term in the preceding expansion multiplied by $a + x$; now we have just seen that $a^{m+1} + Ba^m x$ are the two first terms in the preceding expansion; therefore, by omitting the coefficient of the third term, the value of which we are unable at present to foresee, we shall have

$$(a^{m+1} + Ba^m x) \times (a + x) = a^{m+2} + (B + 1)\, a^{m+1}x + a^m x^2$$

equal to the first and second terms, and the literal part of the third term, in the expansion of $(a+x)^{m+2}$. Increase the exponent of the binomial again by 1, then, in like manner, the expansion of $(a+x)^{m+3}$ will be equal to each term in the preceding expansion multiplied by $a+x$; consequently,

$$(a^{m+2}+a^{m+1}x+a^m x^2)\times(a+x),$$

by omitting the coefficients, $=a^{m+3}+a^{m+2}x+a^{m+1}x^2+a^m x^3=$ the literal parts of the four first terms in the expansion of $(a+x)^{m+3}$: and by thus continually increasing the exponent by 1 to n, we shall have the literal parts of $n+1$ terms in the expansion of

$$(a+x)^{m+n}=a^{m+n}+a^{m+n-1}x+a^{m+n-2}x^2+a^{m+n-3}x^3+\text{ \&c.}$$

where it is plain, that any term is equal to the preceding multiplied by $\frac{x}{a}$.

(126.) Now to determine the coefficients of these terms, it appears that if the coefficient of the second term in the expansion of $(a+x)^{m+1}$ be B, that the coefficient of the second term in the succeeding expansion will be B + 1; therefore the difference between the exponent and second term must be the same in each expansion, whatever be the value of m; but, when $m=0$, the coefficient of the second term in the expansion of $(a+x)^{m+1}$, or $(a+x)^1$ is 1; and the difference between this and the exponent is 0, therefore the difference must be always 0; that is, the coefficient of the second term in any expansion must be equal to the exponent.

(127.) Put now $m+n=r$; and let

$$(a+x)^r=a^r+\text{B}a^{r-1}x+\text{C}a^{r-2}x^2+\text{D}a^{r-3}x^3+\text{ \&c. } \ldots(\text{A})$$

where $\text{B}=r$, and C, D, &c. are undetermined. Square each side of this equation, and we have for the square of the first side $(a^2+2ax+x^2)^r$, or by considering $2ax+x^2$ as one term, it may be written thus, $\{a^2+(2ax+x^2)\}^r$; the quantity within the brackets being a binomial, the first term of which is a^2, and the second $(2ax+x^2)$; therefore, to exhibit the expansion of this, it will be only necessary to write a^2 instead of a, and $(2ax+x^2)$ instead of x in the expansion of $(a+x)^r$, and we have $\{a^2+(2ax+x^2)\}^r=a^{2r}+\text{B}a^{2(r-1)}(2ax+x^2)+\text{C}a^{2(r-2)}(2ax+x^2)^2+\text{D}a^{2(r-3)}(2ax+x^2)^3+$ &c., and by actually involving the quantities within the parentheses, and writing

the terms containing the like powers of x one under the other, the result is

$$\left.\begin{array}{l} a^{2r}+2\text{B}a^{2r-1}x+\ \text{B}a^{2r-2}x^2+4\text{C}a^{2r-3}x^3+\ \&\text{c.} \\ \qquad\qquad\qquad +4\text{C}a^{2r-2}x^2+8\text{D}a^{2r-3}x^3+\ \&\text{c.}\end{array}\right\} \ldots \ldots (\text{A}').$$

Also, for the square of the second side of the equation (A), we have

$$\left.\begin{array}{r} a^{2r}+\text{B}a^{2r-1}x+\ \text{C}a^{2r-2}x^2+\ \text{D}a^{2r-3}x^3+\&\text{c.} \\ \text{B}a^{2r-1}x+\text{B}^2a^{2r-2}x^2+\text{BC}a^{2r-3}x^3+\&\text{c.} \\ \text{C}a^{2r-2}x^2+\text{BC}a^{2r-3}x^3+\&\text{c.} \\ \text{D}a^{2r-3}x^3+\&\text{c.}\end{array}\right\} \ldots \ldots (\text{B}').$$

Now, since these series (A′) and (B′) are equal, whatever be the value of x, by the theorem previously demonstrated, the coefficients of the like powers of x are equal the one to the other; that is

$2\text{B} = 2\text{B}$; or . . . . $\text{B} = \text{B}$;

$$\text{B} + 4\text{C} = 2\text{C} + \text{B}^2, \therefore \text{C} = \frac{\text{B}^2 - \text{B}}{2} = \frac{\text{B}(\text{B}-1)}{2};$$

$$4\text{C} + 8\text{D} = 2\text{D} + 2\text{BC}, \therefore \text{D} = \frac{\text{C}(\text{B}-2)}{3} = \frac{\text{B}(\text{B}-1)(\text{B}-2)}{2\,.\,3}:$$

and, by proceeding in the same manner, we shall find

$$\text{E} = \frac{\text{D}(\text{B}-3)}{4} = \frac{\text{B}(\text{B}-1)(\text{B}-2)(\text{B}-3)}{2\,.\,3\,.\,4}.$$

and also the remaining coefficients F, G, &c. in terms of B; but those already deduced are sufficient to show the law of their formation, since it is obvious that the numerator of each is equal to the numerator of the preceding multiplied by an additional factor; and the denominator equal to the denominator of the preceding multiplied by an additional factor; the factors in the numerator successively decreasing by 1, and those in the denominator successively increasing by 1.

Now B has been shown to be equal to r; hence the expansion of $(a+x)^r$ is

$$(a+x)^r = a^r + ra^{r-1}x + \frac{r(r-1)}{2}a^{r-2}x^2 + \frac{r(r-1)(r-2)}{2\,.\,3}a^{r-3}x^3 +$$

&c.; or restoring the value of $r = m + n$, and putting $m = 0$, we have

1. $(a+x)^n =$

$$a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2\,.\,3}a^{n-3}x^3 + \&c.$$

Putting $m = -(s+n)$, the series becomes

2. $(a+x)^{-s} =$

$$a^{-s} - sa^{-(s+1)}x + \frac{s(s+1)}{2}a^{-(s+2)}x^2 - \frac{s(s+1)(s+2)}{2\,.\,3}a^{-(s+3)}x^3 + \&c.$$

Or, putting $m = \frac{p}{q} - n$, it becomes

3. $(a+x)^{\frac{p}{q}} =$

$$a^{\frac{p}{q}} + \frac{p}{q}a^{\frac{p}{q}-1}x + \frac{\frac{p}{q}(\frac{p}{q}-1)}{2}a^{\frac{p}{q}-2}x^2$$

$$+ \frac{\frac{p}{q}(\frac{p}{q}-1)(\frac{p}{q}-2)}{2\,.\,3}a^{\frac{p}{q}-3}x^3 + \&c.$$

and putting $m = -(\frac{p}{q} + n)$, we have

4. $(a+x)^{-\frac{p}{q}} =$

$$a^{-\frac{p}{q}} - \frac{p}{q}a^{-(\frac{p}{q}+1)}x + \frac{\frac{p}{q}(\frac{p}{q}+1)}{2}a^{-(\frac{p}{q}+2)}x^2$$

$$-\frac{\frac{p}{q}\ (\frac{p}{q}+1)(\frac{p}{q}+2)}{2\,.\,3}a^{-(\frac{p}{q}+3)}x^3 + \&c.$$

which is the law of expansion that we proposed to investigate; the first series showing the expansion when the exponent of the binomial is a *positive* integer; the second showing the expansion when the exponent is a *negative* integer; the third when the exponent is a *positive fraction;* and the fourth when the exponent is a *negative fraction.*

(128.) But our proof of this law extends only to $n+1$ terms; let us therefore examine the coefficients of the respective terms in the first series, with a view to ascertain to how many terms they can possibly extend. These coefficients are

$$1,\ n,\ \frac{n(n-1)}{2},\ \frac{n(n-1)(n-2)}{2 \, . \, 3},\ \frac{n(n-1)(n-2)(n-3)}{2 \, . \, 3 \, . \, 4},\ \&c.$$

Now it is obvious that this series of coefficients can never extend to that in which the numerator is

$$n(n-1)(n-2)(n-3) \ldots . (n-n);$$

that is, in which the negative term in the last factor is n; because $n-n=0$, and, consequently, the coefficient, and, indeed, the entire term would vanish, as well as all that follow it; therefore the series must terminate at the term immediately preceding this; that is, at that in which the negative term in the last factor is $n-1$; now the negative term in the last factor is evidently always 2 less than the number of the term; thus, in the third term it is 1, in the fourth 2, &c., and therefore, when it is $n-1$, the term must be the $n+1$th; consequently, our proof extends throughout the whole series.

(129.) But in the second, and the other two series, the exponents are entirely unconnected with n; these series will, therefore, be unlimited, since $n+1$ terms may express any number from 1 to infinity.

(130.) Hence we conclude that when the exponent is a positive integer, as n, then the series will terminate at the $n+1$th term; but when the exponent is either negative or fractional, the series will not terminate, but may be carried on to infinity; as is also evident from the bare inspection of the terms.

(131.) It also follows, that since in the expansion of $(a+x)^n$, the exponent of a in the first term is n, and that of x, 0; and because the exponent of a in every succeeding term is decreased by 1, and that of x increased by 1, the $n+1$th, or last term, will be x^n, the last but one ax^{n-1}, the last but two a^2x^{n-2}, &c. (the coefficients being omitted), and these will evidently correspond to the literal parts of the first, second, third, &c. terms respectively in the expansion of $(x+a)^n$, and therefore the coefficients of the last, last but one, last but two, &c. terms in the expansion of $(a+x)^n$, are respectively equal to those of the first, second, third, &c. terms in the expansion of $(x+a)^n$, since the series themselves are equal; but the coefficients of the first, second, third, &c. terms in the expansion of $(x+a)^n$ must be the same as those of the corresponding terms in the expansion of $(a+x)^n$, therefore in the expansion of $(a+x)^n$ the

coefficients of the terms from the first to the middle are respectively the same as those from the last to the middle.

(132.) By making a and x each equal to 1, a curious property of the binomial may be exhibited, viz.

$$(1+1)^m, \text{ or } 2^m, = 1 + m + \frac{m(m-1)}{2} +$$

$$\frac{m(m-1)(m-2)}{2\,.\,3} + \frac{m(m-1)(m-2)(m-3)}{2\,.\,3\,.\,4} + \&c.;$$

that is, in any expansion of a binomial, whose terms are both positive, the sum of the coefficients is equal to the same power, or root of 2.

Also, if $a=1$ and $x=-1$, we have the following property, viz.

$$(1-1)^m, \text{ or } 0, = 1 - m +$$

$$\frac{m(m-1)}{2} - \frac{m(m-1)(m-2)}{2\,.\,3} + \frac{m(m-1)(m-2)}{2\,.\,3\,.\,4} - \&c.$$

that is, in any expansion of a binomial, one of whose terms is negative,* the sum of the coefficients is $=0$; and therefore the sum of the positive coefficients must be equal to the sum of the negative ones.

On account of the great importance of the Binomial Theorem in every department of Analysis, we feel disposed to present the student with another method of investigating it. But, not to detain him longer from its practical application, we shall postpone this second mode of establishing it till the close of the Chapter, (see page 173).

APPLICATION OF THE BINOMIAL THEOREM TO THE EXPANSION OF SERIES.

(133.) 1. *To expand $(a+x)^m$ when m is a Positive, or Negative Integer.*

Make the first and second terms a^m, and $ma^{m-1}x$, respectively; then, to find the others, multiply the coefficient of the term last found by the index of a in that term, and the product divided by the

* Binomials of this kind are sometimes called residual quantities.

number of the term will give the coefficient of the next term :* with respect to the literal parts, the powers of a are to decrease, and those of x increase by unity in each successive term. . This will appear plain from inspecting the expansions of $(a+x)^m$ and $(a+x)^{-m}$.

NOTE. When m is positive, the coefficients need only be calculated as far as the middle term, those of the other terms being the same, taken in an inverted order (Art. 131). If one part of the binomial be negative, then the terms involving its odd powers must be negative.

EXAMPLES.

1. It is required to expand $(a+x)^8$, or to express the 8th power of $(a+x)$.

Here

The first term is . . . . a^8.

The second . . . . $8a^7x$.

The third . . . . $\dfrac{8\times 7}{2}a^6x^2 = 28a^6x^2$

The fourth . . . . $\dfrac{28\times 6}{3}a^5x^3 = 56a^5x^3$.

The fifth . . . . $\dfrac{56\times 5}{4}a^4x^4 = 70a^4x^4$.

And from these the other terms are obtained (NOTE).

Hence $(a+x)^8 = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$.

2. It is required to expand $(x-y)^9$, or to find the 9th power of $(x-y)$.

Here

The first term is . . . . x^9.

The second . . . . $-9x^8y$. .

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* It is worthy of notice, that one or other of the two factors forming the product will always be exactly divisible by the number of the term, and that therefore in practice we may first perform the division upon one factor, and then multiply the quotient by the other, which is the simplest method of arriving at the desired coefficient.

The third . . . . $\frac{9 \times 8}{2} x^7y^2 = 36x^7y^2$.

The fourth . . . . $-\frac{36 \times 7}{3} x^6y^3 = -84x^6y^3$.

The fifth . . . . $\frac{84 \times 6}{4} x^5y^4 = 126x^5y^4$.

&c. &c. &c.

Hence $(x-y)^9 = x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$.

3. It is required to expand $(a+x)^{-2}$, or $\frac{1}{(a+x)^2}$.

Here

The first term is . . . . a^{-2}.

The second . . . . $-2a^{-3}x$.

The third . . . . $\frac{-2 \times -3}{2} a^{-4}x^2 = 3a^{-4}x^2$.

The fourth . . . . $\frac{3 \times -4}{3} a^{-5}x^3 = -4a^{-5}x^3$.

&c. &c. &c.

Hence $(a+x)^{-2}$, or $\frac{1}{(a+x)^2}$, $= a^{-2} - 2a^{-3}x + 3a^{-4}x^2 - 4a^{-5}x^3 + \text{\&c.}$, or $\frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \text{\&c.} = \frac{1}{a^2}\left(1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \text{\&c.}\right)$

4. It is required to expand $(a+2x)^{-3}$, or $\frac{1}{(a+2x)^3}$.

Here

The first term is . . . . a^{-3}.

The second . . . . $-3a^{-4}2x = -6a^{-4}x$.

The third . . . . $\frac{-3 \times -4}{2} a^{-5}(2x)^2 = 24a^{-5}x^2$.

The fourth . . . . $\frac{6 \times -5}{3} a^{-6}(2x)^3 = -80a^{-6}x^3$.

&c. &c. &c.

Hence $(a+2x)^{-3}$, or $\dfrac{1}{(a+2x)^3} = a^{-3} - 6a^{-4}x + 24a^{-5}x^2 - 80a^{-6}x^3$ $+$ &c.; or $\dfrac{1}{a^3} - \dfrac{6x}{a^4} + \dfrac{24x^2}{a^5} - \dfrac{80x^3}{a^6} +$ &c. $= \dfrac{1}{a^3}\left(1 - \dfrac{6x}{a} + \dfrac{24x^2}{a^2} - \dfrac{80x^3}{a^3} + \text{\&c.}\right)$

5. It is required to expand $(x-y)^7$, or to find the 7th power of $(x-y)$.

Ans. $\begin{cases} x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - \\ 21x^2y^5 + 7xy^6 - y^7. \end{cases}$

6. It is required to find the 7th power of $(x+2y)$.

Ans. $\begin{cases} x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + \\ 448xy^6 + 128y^7. \end{cases}$

7. It is required to find the cube of $a+b+c$, or of $(a+b)+c$.
Ans. $(a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$, or $a^3 + 3a^2b + 3ab^2$ $+ b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

8. It is required to find the expansion of $\dfrac{2}{(c+x)^2}$.

Ans. $\dfrac{2}{c^2} - \dfrac{4x}{c^3} + \dfrac{6x^2}{c^4} - \dfrac{8x^3}{c^5} +$ &c.

9. It is required to find the expansion of $\dfrac{a^2}{(a+2b)^3}$.

Ans. $\dfrac{1}{a}\left(1 - \dfrac{6b}{a} + \dfrac{24b^2}{a^2} - \dfrac{80b^3}{{}^3} + \text{\&c.}\right.$

(133.) 2. *To expand* $(a+x)^{\frac{m}{n}}$, $\frac{m}{n}$ *being either Positive or Negative.*

Agreeably to the law of the terms already established, if we put Q for $\dfrac{x}{a}$, we shall have

$$(a+x)^{\frac{m}{n}} = \overbrace{a^{\frac{m}{n}}}^{\text{A}} + \overbrace{\frac{m}{n}\text{A}\text{Q}}^{\text{B}} + \overbrace{\frac{\frac{m}{n}-1}{2}\text{B}\text{Q}}^{\text{C}} + \overbrace{\frac{\frac{m}{n}-2}{3}\text{C}\text{Q}}^{\text{D}} + \text{\&c.}$$

where A, B, C, &c. represent the first, second, third, &c. terms respectively.

Or,

$$(a+x)^{\frac{m}{n}} = \overbrace{a^{\frac{m}{n}}}^{A} + \overbrace{\frac{m}{n}AQ}^{B} + \overbrace{\frac{m-n}{2n}BQ}^{C} + \overbrace{\frac{m-2n}{3n}CQ}^{D} + \text{&c.}$$

which last form is the most commodious in practice, and differs but little from that in which the binomial theorem was first proposed by Newton.

EXAMPLES.

1. Express the value of $\sqrt[3]{b^3+x}$ in a series.

Here $(a+x)^{\frac{m}{n}} = (b^3+x)^{\frac{1}{3}}$, $\therefore\ a = b^3$, $m = 1$, $n = 3$, and

$$Q = \frac{x}{b^3}.$$

Whence $a^{\frac{m}{n}} = (b^3)^{\frac{1}{3}} = b = A.$

$$\frac{m}{n}AQ = \tfrac{1}{3}b \times \frac{x}{b^3} = \frac{x}{3b^2} = B.$$

$$\frac{m-n}{2n}BQ = -\tfrac{2}{6} \times \frac{x}{3b^2} \times \frac{x}{b^3} = -\frac{2x^2}{3.6b^5} = C.$$

$$\frac{m-2n}{3n}CQ = -\tfrac{5}{9} \times -\frac{2x^2}{3.6b^5} \times \frac{x}{b^3} = \frac{2.5x^3}{3.6.9b^8} = D.$$

$$\frac{m-3n}{4n}DQ = -\tfrac{8}{12} \times \frac{2.5x^3}{3.6.9b^8} \times \frac{x^3}{b^3} = -\frac{2.5.8x^4}{3.6.9.12b^{11}} = E.$$

Here the law of continuation is manifest;

$$\therefore\ \sqrt[3]{b^3+x} = b + \frac{x}{3b^2} - \frac{2x^2}{3.6b^5} + \frac{2.5x^3}{3.6.9b^8} - \frac{2.5.8x^4}{3.6.9.12b^{11}} + \text{&c.}$$

2. Find the value of $\frac{1}{(b^2+x)^{\frac{1}{2}}}$ in a series.

Here $\frac{1}{(b^2+x)^{\frac{1}{2}}} = (b^2+x)^{-\frac{1}{2}}$, $\therefore\ a = b^2$, $m = -1$, $n = 2$,

$$\text{and } Q = \frac{x}{b^2}.$$

Whence $a^{\frac{m}{n}} = (b^2)^{-\frac{1}{2}} = \frac{1}{b} = \text{A}.$

$$\frac{m}{n}\text{AQ} = -\frac{1}{2b} \times \frac{x}{b^2} = -\frac{x}{2b^3} = \text{B}.$$

$$\frac{m-n}{2n}\text{BQ} = -\tfrac{3}{4} \times -\frac{x}{2b^3} \times \frac{x}{b^2} = \frac{3x^2}{2.4b^5} = \text{C}.$$

$$\frac{m-2n}{3n}\text{CQ} = -\tfrac{5}{6} \times \frac{3.x^2}{2.4b^5} \times \frac{x}{b^2} = -\frac{3.5x^3}{2.4.6b^7} = \text{D}.$$

$$\frac{m-3n}{4n}\text{DQ} = -\tfrac{7}{8} \times -\frac{3.5x^3}{2.4.6b^7} \times \frac{x}{b^2} = \frac{3.5.7x^4}{2.4.6.8b^9} = \text{E}.$$

&c. &c. &c.

$$\therefore \frac{1}{(b^2+x)^{\frac{1}{2}}} = \frac{1}{b} - \frac{x}{2b^3} + \frac{3x^2}{2.4b^5} - \frac{3.5x^3}{2.4.6b^7} + \frac{3.5.7x^4}{2.4.6.8b^9} - \&c.$$

3. Find the value of $(b^2 - x)^{\frac{1}{2}}$ in a series.

Here $a = b^2$, $x = -x$, $m = 1$, $n = 2$, and $\text{Q} = -\frac{x}{b^2}$.

Whence $a^{\frac{m}{n}} = (b^2)^{\frac{1}{2}} = b = \text{A}.$

$$\frac{m}{n}\text{AQ} = \tfrac{1}{2}b \times -\frac{x}{b^2} = -\frac{x}{2b} = \text{B}.$$

$$\frac{m-n}{2n}\text{BQ} = -\tfrac{1}{4} \times -\frac{x}{2b} \times -\frac{x}{b^2} = -\frac{x^2}{2.4b^3} = \text{C}.$$

$$\frac{m-2n}{3n}\text{CQ} = -\tfrac{3}{6} \times -\frac{x^2}{2.4b^3} \times -\frac{x}{b^2} = -\frac{3x^3}{2.4.6b^5} = \text{D}.$$

$$\frac{m-3n}{4n}\text{DQ} = -\tfrac{5}{8} \times -\frac{3x^3}{2.4.6b^5} \times -\frac{x}{b^2} = -\frac{3.5x^4}{2.4.6.8b^7} = \text{E}.$$

&c. &c. &c.

$$\therefore (b^2 - x)^{\frac{1}{2}} = b - \frac{x}{2b} - \frac{x^2}{2.4b^3} - \frac{3x^3}{2.4.6b^5} - \frac{3.5x^4}{2.4.6.8b^7} - \&c.$$

4. Find the value of $(b^2 - x)^{\frac{3}{4}}$ in a series.

Here $a = b^2$, $x = -x$, $m = 3$, $n = 4$, and $\text{Q} = -\frac{x}{b^2}$.

Whence $a^{\frac{m}{n}} = (b^2)^{\frac{3}{4}} = b^{\frac{3}{2}} = \text{A}.$

$$\frac{m}{n}\text{AQ} = \tfrac{3}{4}b^{\frac{3}{2}} \times -\frac{x}{b^2} = -\frac{3x}{4b^{\frac{1}{2}}} = \text{B}.$$

$$\frac{m-n}{2n}\text{BQ} = -\tfrac{1}{8} \times -\frac{3x}{4b^{\frac{1}{2}}} \times -\frac{x}{b^2} = -\frac{3x^2}{4.8b^{\frac{5}{2}}} = \text{C}.$$

$$\frac{m-2n}{3n}\text{CQ} = -\tfrac{5}{12} \times -\frac{3x^2}{4.8b^{\frac{5}{2}}} \times -\frac{x}{b^2} = -\frac{3.5x^3}{4.8.12b^{\frac{9}{2}}} = \text{D}.$$

$$\frac{m-3n}{4n}\text{DQ} = -\tfrac{9}{16} \times -\frac{3.5x^3}{4.8.12b^{\frac{9}{2}}} \times -\frac{x}{b^2} = -\frac{3.5.9x^4}{4.8.12.16b^{\frac{13}{2}}} = \text{E}.$$

&c. &c. &c.

$$\therefore (b^2-x)^{\frac{3}{4}} = b^{\frac{3}{2}} - \frac{3x}{4b^{\frac{1}{2}}} - \frac{3x^2}{4.8b^{\frac{5}{2}}} - \frac{3.5x^3}{4.8.12b^{\frac{9}{2}}} - \frac{3.5.9x^4}{4.8.12.16b^{\frac{13}{2}}} -$$

&c. Or,

$$(b^2-x)^{\frac{3}{4}} = \frac{1}{\sqrt{b}}\left(b^2 - \frac{3x}{2^2} - \frac{3x^2}{2^5b^2} - \frac{5x^3}{2^7b^4} - \frac{5.9x^4}{2^{11}b^6} - \text{\&c.}\right)$$

5. Express the value of $\sqrt[3]{7}$ in a series.

Here $\sqrt[3]{7} = (8-1)^{\frac{1}{3}}$, $\therefore a = 8$, $x = -1$, $m = 1$, $n = 3$, and

$$\text{Q} = -\tfrac{1}{8} = -\frac{1}{2^3}.$$

Whence $a^{\frac{m}{n}} = 8^{\frac{1}{3}} = 2 = \text{A}.$

$$\frac{m}{n}\text{AQ} = \tfrac{1}{3} \times 2 \times -\frac{1}{2^3} = -\frac{1}{3.2^2} = \text{B}.$$

$$\frac{m-n}{2n}\text{BQ} = -\tfrac{2}{6} \times -\frac{1}{3.2^2} \times -\frac{1}{2^3} = -\frac{1}{3.6.2^4} = \text{C}.$$

$$\frac{m-2n}{3n}\text{CQ} = -\tfrac{5}{9} \times -\frac{1}{3.6.2^4} \times -\frac{1}{2^3} = -\frac{5}{3.6.9.2^7} = \text{D}.$$

$$\frac{m-3n}{4n}\text{DQ} = -\tfrac{8}{12} \times -\frac{5}{3.6.9.2^7} \times -\frac{1}{2^3} = -\frac{5.8}{3.6.9.12.2^{10}} = \text{E}.$$

&c. &c. &c.

$$\therefore \sqrt[3]{7} = 2 - \frac{1}{3.2^2} - \frac{1}{3.6.2^4} - \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} - \text{\&c.}$$

6. Required the value of $\sqrt{b^2+x}$ in a series.

Ans. $b + \frac{x}{2b} - \frac{x^2}{2.4b^3} + \frac{3x^3}{2.4.6b^5} - \frac{3.5x^4}{2.4.6.8b^7} + \text{\&c.}$

7. Required the value of $\dfrac{c^2}{(c^2-x)^{\frac{1}{2}}}$ in a series.

$$\text{Ans. } c+\frac{x}{2c}+\frac{3x^2}{2.4c^3}+\frac{3.5x^3}{2.4.6c^5}+\frac{3.5.7x^4}{2.4.6.8c^7}+\ \&c.$$

8. Required the value of $(a+x)^{\frac{2}{3}}$ in a series.

$$\text{Ans. } a^{\frac{2}{3}}(1+\frac{2x}{3a}-\frac{x^2}{3^2a^2}+\frac{4x^3}{3^4a^3}-\frac{7x^4}{3^5a^4}+\ \&c.)$$

9. Required the value of $\sqrt[3]{9}$ in a series.

$$\text{Ans. } 2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\ \&c.$$

10. Required the value of $\sqrt{2}$ in a series.

$$\text{Ans. } 1+\tfrac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\ \&c.$$

11. Required the value of $(a^2-x^2)^{\frac{3}{4}}$ in a series.

$$\text{Ans. } \frac{1}{\sqrt{a}}(a^2-\frac{3x^2}{2^2}-\frac{3x^4}{2^5a^2}-\frac{5x^6}{2^7a^4}-\frac{5.9x^8}{2^{11}a^6}-\ \&c.)$$

(135.) We promised, at page 166, to present the student with another method of investigating the Binomial Theorem. The method which we had in view is that which follows.

It has already been shown (page 161,) that the first term in the developement of $(a+x)^{\frac{m}{n}}$ must always be $a^{\frac{m}{n}}$: we may assume, therefore,

$$(a+x)^{\frac{m}{n}}=a^{\frac{m}{n}}+Bx+Cx^2+Dx^3+\ \&c.$$

or, changing x into y,

$$(a+y)^{\frac{m}{n}}=a^{\frac{m}{n}}+By+Cy^2+Dy^3+\ \&c.$$

hence, by subtraction,

$$(a+x)^{\frac{m}{n}}-(a+y)^{\frac{m}{n}}=B(x-y)+C(x^2-y^2)+D(x^3-y^3)+\ \&c.$$

and consequently,

$$\frac{(a+x)^{\frac{m}{n}}-(a+y)^{\frac{m}{n}}}{(a+x)-(a+y)}=\frac{(a+x)^{\frac{m}{n}}-(a+y)^{\frac{m}{n}}}{x-y}=\mathrm{B}+\mathrm{C}(x+y)+$$
$$\mathrm{D}(x^2+yx+y^2)+\mathrm{E}(x^3+yx^2+y^2x+y^3)+\&c.$$

If now we were to suppose $x=y$, the second member of this equation would present itself in a definite and intelligible form; but the first would become $\frac{0}{0}$, a fraction in which both numerator and denominator have vanished. As however this vanishing fraction has a definite value, shown by the second side of the equation, there can be no doubt that its ambiguous form, $\frac{0}{0}$, must have arisen from some common factor in the numerator and denominator of the original fraction having become 0, by putting in that fraction $x=y$.* If then we could discover this common factor, we should be able, by expunging it from both numerator and denominator, to free the fraction from all ambiguity, and the result of our hypothesis, $x=y$, would then be definite in *form* as well as in value. Now we shall be able to effect this by transforming our fraction into another of equivalent value, by means of the following substitutions.

Put $u=(a+x)^{\frac{1}{n}}$, $v=(a+y)^{\frac{1}{n}}$ $\therefore$ $x-y=u^n-v^n$, and, consequently

$$\frac{u^m-v^m}{u^n-v^n}=\mathrm{B}+\mathrm{C}(x+y)+\mathrm{D}(x^2+yx+y^2)+$$
$$\mathrm{E}(x^3+yx^2+y^2x+y^3)+\&c.$$

Now both numerator and denominator of the first member of this equation are divisible by $u-v$, and $u-v$ is the very factor which vanishes for $x=y$, as is at once seen by referring to the substitutions just proposed. This factor will be removed by actually dividing numerator and denominator by $u-v$, which reduces the fraction to

$$\frac{u^{m-1}+vu^{m-2}+v^2u^{m-3}+\ldots\ldots v^{m-1}}{u^{n-1}+vu^{n-2}+v^2u^{n-3}+\ldots\ldots v^{n-1}}=\mathrm{B}+\mathrm{C}(x+y)+$$
$$\mathrm{D}(x^2+yx+y^2)+\&c.$$

---

* The theory of *vanishing fractions* will be found fully discussed in the volume on the *Theory of Equations*.

Introducing now the proposed hypothesis, $x=y$, which leads to $v=u$, we have

$$\frac{mv^{m-1}}{nv^{n-1}} = \frac{mv^m}{nv^n} = B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.;$$

that is, by restoring the value of v,

$$\frac{m}{n} \cdot \frac{(a+x)^{\frac{m}{n}}}{a+x} = B + 2Cx + 3Dx^2 + 4Ex^3 + \&c.$$

Multiply both members by $a + x$, and then, instead of $(a + x)^{\frac{m}{n}}$ in the first member, write its developed form with which we set out, and we shall have

$$\frac{m}{n}a^{\frac{m}{n}} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \&c. =$$

$$Ba + \left.\begin{matrix}2Ca\\B\end{matrix}\right| x + \left.\begin{matrix}3Da\\2C\end{matrix}\right| x^2 + \left.\begin{matrix}4Ea\\3D\end{matrix}\right| x^3 + \&c.$$

Hence, by the theorem at page 160,

$$Ba = \frac{m}{n}a^{\frac{m}{n}}, \text{ therefore } B = \frac{m}{n}a^{\frac{m}{n}-1}$$

$$2Ca + B = \frac{m}{n}B \quad . \; . \; . \; . \; . \quad C = \frac{(\frac{m}{n}-1)B}{2a}$$

$$3Da + 2C = \frac{m}{n}C \quad . \; . \; . \; . \; . \quad D = \frac{(\frac{m}{n}-2)C}{3a}$$

$$4Ea + 3D = \frac{m}{n}D \quad . \; . \; . \; . \; . \quad E = \frac{(\frac{m}{n}-3)D}{4a},$$

$$\&c. \qquad\qquad\qquad \&c.$$

Consequently,

$$(a+x)^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n}a^{\frac{m}{n}-1}x + \frac{\frac{m}{n}(\frac{m}{n}-1)}{2}a^{\frac{m}{n}-2}x^2 +$$

$$\frac{\frac{m}{n}(\frac{m}{n}-1)(\frac{m}{n}-2)}{2 \, . \, 3}a^{\frac{m}{n}-3}x^3 + \&c.$$

In this demonstration m and n may obviously be any whole numbers whatever, and m may be either positive or negative.

(136.) By the aid of the binomial theorem, we may, by a simple and elegant process, obtain the development of a^x in a series according to the ascending powers of x. The quantity a^x is called an *exponential* quantity, and the development of which we speak is called the *exponential theorem:* this theorem we shall now investigate, on account of its importance in the theory of logarithms, and in other departments of analysis.

The Exponential Theorem.

We are here required to exhibit the development of a^x according to the ascending powers of x. We shall commence by showing that the proposed form of development is possible.

Put $a = 1 + b$ $\therefore$ $a^x = (1 + b)^x$, and, by the binomial theorem,

$$(1 + b)^x = 1 + xb + \frac{x(x-1)}{2} b^2 + \frac{x(x-1)(x-2)}{2 \cdot 3} b^3$$
$$+ \frac{x(x-1)(x-2)(x-3)}{2 \cdot 3 \cdot 4} b^4 + \&c.$$

and it is obvious, that if the multiplications indicated by the numerators in the right hand member of this equation were actually executed, the result would be a series of monomials in x, in x^2, in x^3, &c., which we might arrange in a regular ascending order. The term in x is ascertainable at once from mere inspection: it is $\{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.\}x$; so that we may safely conclude that a^x may be developed in the form

$$a^x = 1 + \{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.\}x + Bx^2 + Cx^3 + \&c.$$

Having thus seen the possibility of the proposed development, let us assume

$$\left.\begin{array}{l} a^x = 1 + Ax + Bx^2 + Cx^3 + \&c. \\ \text{in like manner, } a^y = 1 + Ay + By^2 + Cy^3 + \&c. \end{array}\right\} \ldots . (1);$$

$\therefore$ by subtraction,

$$a^x - a^y = A(x - y) + B(x^2 - y^2) + C(x^3 - y^3) + \&c. \ldots . (2).$$

Again, by the original assumption (1),

$$a^{(x-y)} = 1 + A(x-y) + B(x-y)^2 + C(x-y)^3 + \&c.,$$

or transposing the 1, and then multiplying each member by a^y,

$$a^x - a^y = a^y \{A(x-y) + B(x-y)^2 \times C(x-y)^3 + \&c.\} \ . \ . \ . \ (3).$$

Hence the second members of (2) and (3) are equal; these we may simplify by dividing each by $x-y$. Perform this division, and in the quotients put $x=y$, then the second member of (2) will become $A + 2Bx + 3Cx^2 + \&c.$, and that of (3) will be reduced to simply $A \cdot a^y$, or, which is the same thing, to $A \cdot a^x$; hence, substituting for a^x the series which we have assumed for its development, we have this equation for determining the assumed coefficients, viz.

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. = A(1 + Ax + Bx^2 + Cx^3 + \&c.);$$

hence, comparing the coefficients of the like powers of x, we have

$$B = \frac{A^2}{2},\ C = \frac{A^3}{2 \cdot 3},\ D = \frac{A^4}{2 \cdot 3 \cdot 4},\ \&c.$$

Hence (1),

$$a^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2 \cdot 3} + \frac{A^4x^4}{2 \cdot 3 \cdot 4} + \&c.,$$

which is the *exponential theorem*, and in which

$$A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

CHAPTER VI.

ON LOGARITHMS AND THEIR APPLICATIONS.

(137.) LOGARITHMS are certain numbers invented by Lord Napier for the purpose of facilitating arithmetical computations by reducing every numerical process to the simple operations of addition and subtraction. To understand how this is effected, we must consider every positive number as a power, whole or fractional, of some assumed root fixed upon at pleasure; from this root all positive numbers are supposed to be generated, by involution or evolution, and it is the *exponent* of this root which is called the *logarithm* of the number or power generated. A table therefore containing the logarithms of the numbers 1, 2, 3, 4, &c. is nothing more than a table of the several exponents which the assumed root must take to produce the numbers 1, 2, 3, 4, &c. Thus, if a be any assumed number, and such values be successively given to x that will make $a^x = b$, $a^x = c$, $a^x = d$, &c., then these different values of x are the logarithms of b, c, d, &c. respectively: If $x = 0$, then $a^x = 1$, whatever be the value of a, (Art. 38, Chap. I.); hence the logarithm of 1 is always 0.

(138.) The assumed root a is called the *base* of the system of logarithms, and from different bases different systems of logarithms must evidently arise; but it has been found to be most convenient to assume 10 for the base, and upon this assumption all our modern logarithmic tables are constructed. The advantage of the base 10 over every other base will be seen hereafter.

(139.) Assuming therefore $a = 10$, we have

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \text{ \&c.}$$

that is, the log. of 1 is 0, the log. of 10 is 1, the log. of 100 is 2, the log. of 1000 is 3, &c.

Also $10^{-1} = \frac{1}{10}$, $10^{-2} = \frac{1}{100}$, $10^{-3} = \frac{1}{1000}$, &c.;

that is, the log. of $\frac{1}{10}$ is -1, the log. of $\frac{1}{100}$ is -2, the log. of $\frac{1}{1000}$ is -3, &c.

(140.) Hence, since the log. of 1 is 0, and the log. of 10, 1, it follows that the log. of any number between 0 and 10 must lie between 0 and 1; and in the same manner the log. of any number between 10 and 100 must lie between 1 and 2, &c., and therefore these logarithms may be either accurately found, or may be approximated to, to any degree of precision. But before we explain the method of obtaining this approximate value of the logarithm of any given number, it will be convenient to establish the following characteristic properties of logarithms.

(141.) THEOREM 1. The sum of the logarithms of any two numbers is equal to the logarithm of their product.

Let b be any number, and let its logarithm be x; and let c be any other number, whose logarithm is x'; then $a^x = b$, and $a^{x'} = c$; and by multiplying, $a^{x+x'} = bc$; that is, $x + x'$ is the logarithm of bc.

Cor. 1. Hence the sum of the logarithms of any number of numbers is the logarithm of their product.

Cor. 2. Therefore n times the logarithm of any number is the logarithm of its nth power.

THEOREM 2. The difference of the logarithms of any two numbers is equal to the logarithm of their quotient.

For since $a^x = b$, and $a^{x'} = c$, by dividing,

$$\frac{a^x}{a^{x'}} = a^{x-x'} = \frac{b}{c}, \text{ that is, } x - x' = \log. \frac{b}{c}.$$

THEOREM 3. The nth part of the logarithm of any number is equal to the logarithm of its nth root.

For if $a^x = b$, $a^{\frac{x}{n}} = b^{\frac{1}{n}}$, that is, $\frac{x}{n} = \log. b^{\frac{1}{n}}$.

THEOREM 4. If there be any series of quantities in geometrical progression, their logarithms will be in arithmetical progression.

Let the geometrical progression be b, nb, n^2b, n^3b, &c., and let x be the log. of b, and z the log. of n; then $a^x = b$, and $a^z = n$, therefore the progression is the same as

$$a^x,\ a^{x+z},\ a^{x+2z},\ a^{x+3z},\ \&c.,$$

where the logarithms x, $x + z$, $x + 2z$, $x + 3z$, &c. are in arithmetical progression.

PROBLEM.

(142.) To find the logarithm of any given number.

Let N be any given number, then it is required to find the value of x in terms of a and N, so that we may have $a^x = \text{N}$. For this purpose put $a = 1 + m$, and $\text{N} = 1 + n$; then $(1 + m)^x = 1 + n$, and therefore $(1 + m)^{xy} = (1 + n)^y$, whatever be the value of y; hence, by expansion,

$$1 + xym + \frac{xy(xy-1)}{2}m^2 + \frac{xy(xy-1)(xy-2)}{2 \cdot 3}m^3 + \text{\&c.} =$$

$$1 + yn + \frac{y(y-1)}{2}n^2 + \frac{y(y-1)(y-2)}{2 \cdot 3}n^3 + \text{\&c.};$$

or expunging the 1, and dividing by y, we have

$$x(m + \frac{xy-1}{2}m^2 + \frac{(xy-1)(xy-2)}{2 \cdot 3}m^3 + \text{\&c.}) =$$

$$n + \frac{y-1}{2}n^2 + \frac{(y-1)(y-2)}{2 \cdot 3}n^3 + \text{\&c.}$$

Suppose now $y = 0$, and this equation becomes

$$x(m - \frac{m^2}{2} + \frac{m^3}{3} - \text{\&c.}) =$$

$$n - \frac{n^2}{2} + \frac{n^3}{3} - \text{\&c.}$$

whence $x = \log.(1 + n) = \dfrac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \text{\&c.}}{m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \text{\&c.}}$;

or substituting for n and m, their respective values $\text{N} - 1$, and $a - 1$, we have

$$\log. \text{N} = \frac{(\text{N}-1) - \frac{1}{2}(\text{N}-1)^2 + \frac{1}{3}(\text{N}-1)^3 - \text{\&c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}}$$

Hence we have the value of log. N in terms of N and a; but this expression for the logarithm of any number is of but little use in constructing a table of logarithms: we must therefore investigate a method of finding other expressions that may be more suitable for this purpose.

(143.) Since the value of the denominator of the above fraction

depends entirely upon the value of the base a, it will accordingly differ in different systems of logarithms; but the numerator being wholly independent of the base a, must be the same in every system.

(144.) The reciprocal of the denominator is called the *modulus* of the system, and is usually represented by M; so that we have

$$\log.(1+n) = M(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&c.);$$

and supposing n negative,

$$\log.(1-n) = M(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \&c.);$$

and subtracting this equation from the former,

$$\log.(1+n) - \log.(1-n) = \log.\frac{1+n}{1-n}\ (\text{theo. } 2) =$$

$$2M(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.)$$

(145.) Now, since this is true for every value of n, put

$$\frac{1}{2P+1} = n, \text{ then}$$

$$1+n = \frac{2P+2}{2P+1}, \text{ and } 1-n = \frac{2P}{2P+1}, \therefore \frac{1+n}{1-n} = \frac{P+1}{P};$$

consequently,

$$\log.(P+1) - \log.P = 2M\left(\frac{1}{2P+1} + \frac{1}{3(2P+1)^3} + \frac{1}{5(2P+1)^5} + \&c.;\right.$$

$$\therefore \log.(P+1) = 2M\left(\frac{1}{2P+1} + \frac{1}{3(2P+1)^3} + \frac{1}{5(2P+1)^5} + \&c.\right) + \log.P.$$

Hence, if log. P be given, the log. of the next greater number may be found by this series, which converges* very fast, and therefore, since the log. 1 is given $= 0$, we can from this get log. 2, and thence the logs. of all the natural numbers in succession.

A series is said to converge when its value is finite; its terms diminish in such a way that the more of them we take, setting out from the first, the nearer will their sum approach to that of the entire series. The more rapid the rate of diminution is, the greater is the convergency of the series, that is, the less will any proposed number of the leading terms differ from the whole sum. (See the new edition of the "Essay on Logarithms.")

(146.) *To construct a Table of Napierian, or Hyperbolic Logarithms.*

Before we can employ the series which we have just given for the purpose of forming a table, we must assign some value to M, and as this value may be arbitrary, let it be 1, which is the value assumed by Napier, the inventor of logarithms; we shall then have

$$\log.(P+1)=2\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\&c.\right)+\log.P;$$

and making P successively equal to 1, 2, 3, &c., we shall have

$$\log.\ 2=2\left(\tfrac{1}{3}+\frac{1}{3^4}+\frac{1}{5\,.\,3^5}+\&c.\right)\ \ldots\ldots\ =\ \cdot6931472$$

$$\log.\ 3=2\left(\tfrac{1}{5}+\frac{1}{3\,.\,5^3}+\frac{1}{5^6}+\&c.\right)+\log.2\ \ldots\ =1\cdot0986123$$

$$\log.\ 4=2\log.2\ (\text{theo. } 1)\ \ldots\ldots\ =1\cdot3862944$$

$$\log.\ 5=2\left(\tfrac{1}{9}+\frac{1}{3\,.\,9^3}+\frac{1}{5\,.\,9^5}+\&c.\right)+\log.4\ \ldots\ =1\cdot6094379$$

$$\log.\ 6=\log.2+\log.3\ \ldots\ldots\ =1\cdot7917595$$

$$\log.\ 7=2\left(\tfrac{1}{13}+\frac{1}{3(13)^3}+\frac{1}{5(13)^5}+\&c.\right)+\log.6\ =1\cdot9459101$$

$$\log.\ 8=3\log.2\ \ldots\ldots\ =2\cdot0794415$$

$$\log.\ 9=2\log.3\ \ldots\ldots\ =2.1972246$$

$$\log.10=\log.2+\log.5\ \ldots\ldots\ =2\cdot3025851$$

&c. &c.

By proceeding in this way, the logarithms of all the natural numbers according to this particular system may be obtained; but tables constructed conformably to this system, in which we see the logarithm of 10 is 2·3025851, are by no means so advantageous for the general purposes of computation as those in which the logarithm of 10 is 1, as has been before observed; we shall therefore show how

(147.) *To construct a Table of Common Logarithms.*

In the system of common logarithms, the value of M is to be determined from the supposition that the base a is 10, and as the value of the logarithms in any system depends entirely on the value of 2M,

if this value in one system be r times that in another, the logarithm of any number by the former system must be r times that by the latter, and *vice versa;* now, in the hyperbolic system, the logarithm of 10 is 2.3025851, therefore, in order that the logarithm of 10 may be 1, the value of 2M, in the common system, must be the 2·3025851th part of its value in the hyperbolic system; but in this system $2\text{M}=2$, therefore, in the common system, $2\text{M} = \dfrac{2}{2\cdot3025851} = \cdot86858896$; hence, to construct a table of common logarithms, we have

$$\log.\ (\text{P}+1) = \cdot86858896\left(\frac{1}{2\text{P}+1} + \frac{1}{3(2\text{P}+1)^3} + \frac{1}{5(2\text{P}+1)^5} + \right.$$
$$\left.\&c.\right) + \log.\ \text{P};$$

that is, by making $\text{P} = 1, 2, 3$, &c. successively,

$$\log.\ 2 = \cdot86858896\left(\tfrac{1}{3} + \frac{1}{3^4} + \frac{1}{5\,.\,3^5} + \&c.\right)\ .\ .\ = \cdot3010300$$

$$\log.\ 3 = \cdot86858896\left(\tfrac{1}{5} + \frac{1}{3\,.\,5^3} + \frac{1}{5^6} + \&c.\right) + \log.2 = \cdot4771213$$

$$\log.\ 4 = 2\log.\ 2\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . = \cdot6020600$$

$$\log.\ 5 = \cdot86858896\left(\tfrac{1}{9} + \frac{1}{3\,.\,9^3} + \frac{1}{5\,.\,9^5} + \&c.\right) + \log.4 = \cdot6989700$$

$$\log.\ 6 = \log.\ 2. + \log.\ 3\ .\ .\ .\ .\ .\ .\ .\ .\ . = \cdot7781513$$

$$\log.\ 7 = \cdot86858896\left(\tfrac{1}{13} + \frac{1}{3\,(13)^3} + \frac{1}{5\,(13)^5} + \&c.\right) + \log.\ 6.$$
$$= \cdot8450980$$

$$\log.\ 8 = 3\log.\ 2\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . = \cdot9030900$$

$$\log.\ 9 = 2\log.\ 3\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . = \cdot9542426$$

$$\log.\ 10 = \log.\ 2 + \log.\ 5\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . = 1\cdot0000000$$

and in this manner may a table of common logarithms be constructed; and since the logarithms in the hyperbolic system are 2·3025851 times those in the common system, from having a table of the one we may form from it a table of the other.

(148.) In common logarithmic tables, the decimals only of the logarithms are inserted, and the integral part, which is called the *index*, or *characteristic*, is omitted, because this integral part is always known from the number itself, whose logarithm is sought;

for if this number consist of two integral figures, it must be either 10, or some number between 10 and 100, and, consequently, its logarithm must be either 1, or some number between 1 and 2, that is, the integral part must be 1. In the same manner, if the number consist of three integers, the integral part of its logarithm must be 2, &c., so that the index, or characteristic, is always equal to the number of integral figures in the proposed number, *minus* 1.

(149.) It also follows, that in this system the logarithm of any number, and that of one 10 times as great, differ only in the index, the decimal part being the same; so that the decimal parts of the logarithms of all numbers consisting of the same figures remain the same, whether those figures are integers or decimals, or partly integral and partly decimal; thus:

$$\log.\ 3526 \ldots\ldots = 3{\cdot}5472823$$

$$\log.\ 352{\cdot}6 = \log.\ \frac{3526}{10} = \log.\ 3526 - 1 = 2{\cdot}5472823$$

$$\log.\ 35.26 = \log.\ \frac{352{\cdot}6}{10} = \log.\ 352{\cdot}6 - 1 = 1{\cdot}5472823$$

$$\log.\ 3{\cdot}526 = \log.\ \frac{35.26}{10} = \log.\ 35{\cdot}26 - 1 = {\cdot}5472823$$

$$\log.\ {\cdot}3526 = \log.\ \frac{3{\cdot}526}{10} = \log.\ 3{\cdot}526 - 1 = \bar{1}{\cdot}5472823$$

$$\log.\ {\cdot}03526 = \log.\ \frac{{\cdot}3526}{10} = \log.\ {\cdot}3526 - 1 = \bar{2}{\cdot}5472823$$

$$\log.\ {\cdot}003526 = \log.\ \frac{{\cdot}03526}{10} = \log.\ {\cdot}03526 - 1 = \bar{3}{\cdot}5472823$$

&c. &c.

Also,

$$\log.\ \frac{1}{3526} \ldots\ldots = -3{\cdot}5472823$$

$$\log.\ \frac{1}{352{\cdot}6} \ldots\ldots = -2{\cdot}5472823$$

$$\log.\ \frac{1}{35{\cdot}26} \ldots\ldots = -1{\cdot}5472823$$

&c. &c.

We may now perceive the superiority of this system above every other, since the above property, which belongs only to this particular system, will evidently greatly facilitate the construction of a table, it being only necessary to compute the logarithms of the whole numbers; whereas, in every other system, each particular number, whether integral or decimal, requires a particular logarithm.

These advantages of the present system were first suggested by Mr. Briggs, soon after the invention of logarithms, and on this account are sometimes called Briggs's logarithms.

To determine the Napierian Base.

(150.) We have already remarked, that, in Napier's system, the base was that particular value of a which satisfied the condition

$$(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.=1.$$

Let us call this particular value e, then, by the exponential theorem, (p. 176),

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\,.\,3}+\frac{x^4}{2\,.\,3\,.\,4}+\&c.$$

which for $x=1$, gives for the base e the value

$$e=1+1+\frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\&c.$$

which may be thus calculated:

$$\begin{array}{rcrl}
 & & 2. \quad . \quad . \quad . & = a \\
\tfrac{1}{2} & = & \cdot 5 \quad . \quad . \quad . & = b \\
\tfrac{1}{3}b & = & \cdot 1666666666 & = c \\
\tfrac{1}{4}c & = & 416666666 & = d \\
\tfrac{1}{5}d & = & 83333333 & = e \\
\tfrac{1}{6}e & = & 13888888 & = f \\
\tfrac{1}{7}f & = & 1984127 & = g \\
\tfrac{1}{8}g & = & 248016 & = h \\
\tfrac{1}{9}h & = & 27557 & = i \\
\tfrac{1}{10}i & = & 2755 & = k \\
\tfrac{1}{11}k & = & 250 & = l \\
\tfrac{1}{12}l & = & 21 & = m \\
\hline
 & & 2\cdot 718281828 & \\
\hline
\end{array}$$

hence the value of the Naperian or hyperbolic base is 2·718281828.

What is here said upon the subject of logarithms is doubtless sufficient to convey to the student a correct notion of their nature and properties, as also of the practicability of constructing a table of them to any extent. The labour, however, of actually computing a whole table of logarithms by means of the series here investigated, would be great in the extreme; they are, however, susceptible of a variety of transformations much better adapted to the use of the computer. To explain and exhibit these would carry us too far into the business of series, and would occupy too large a portion of this treatise. But the inquiring student, who is desirous of ample information upon the most expeditious methods of calculating a table of logarithms, may refer to the second edition of the author's "Essay on the Computation of Logarithms;" and the manner of using a table thus constructed is fully explained, in the introduction prefixed to the "Mathematical Tables."

APPLICATION OF LOGARITHMS.

LOGARITHMICAL ARITHMETIC.

(151.) From what has been already said on the nature and properties of logarithms, the following operations, performed by means of a table, will be readily understood without any further explanation.

EXAMPLE 1. Multiply 23·14 by 5·062.

Here the log. of 23·14 in the tables* is 1·3643634
log. of 5·062 . . . . . . ·7043221

2·0686855 = log. of 117·1347 = the product.

* The tables employed are Young's "Mathematical Tables," computed to seven places of decimals.

2. Divide ·06314 by ·007241.

Here the log. of ·06314 is $\bar{2}$·8003046
log. of ·007241 $\bar{3}$·8597985

·9405061 = log. of. 8·719792 = the quotient.

Required the fourth power of ·09163.

Here the log. of ·09163 is $\bar{2}$·9620377
4

$\bar{5}$·8481508 = log. of ·0000704938.

Required the tenth root of 2.

Here $\frac{\text{log. } 2}{10} = \frac{0\cdot30103}{10} = \cdot030103 = \text{log. } 1\cdot07179.$

Required the value of $\frac{8^5 \times \sqrt[3]{7}}{\sqrt[5]{6}}$.

Here 5 log. 8 + $\frac{1}{3}$ log. 7 — $\frac{1}{5}$ log. 6 = 4·51545 + ·2816993 — ·1556302 = 4.6415191 = log. 43794·53.

Required the value of $\frac{24^6 \times \sqrt[3]{17}}{4821 \times 6^4}$.

Ans. 78·64561.

required the value of $\sqrt{\frac{284\sqrt[3]{621}}{43^6}}$.

Ans. $\frac{1}{5\cdot72835}$.

The few examples here given are sufficient to show the great advantage of logarithms in abridging arithmetical labour, in which indeed consists their principal, although not their only value. There are many analytical researches which it would be impossible to carry on without their aid, and many others in which the introduction of logarithmic formulas greatly facilitates the deductive process. It would be easy to propose questions, the solutions of which might be comprised in a few lines by logarithms, but which, without their aid,

would occupy many volumes of closely printed pages. The following is a striking example.

Let there be a series of numbers commencing with 2, and such that each is the square of that which immediately precedes it: it is required to determine the number of figures which the 25th term would consist of.

The series proposed is obviously

$$2,\ 2^2,\ 2^4,\ 2^8,\ 2^{16},\ \&c.$$

the exponent of the nth term being 2^{n-1}, and consequently the exponent of the 25th term is $2^{24} = 16777216$; consequently, calling the 25th term x, we have

$$\begin{aligned} x = 2^{16777216}, \text{ whence } \log. x &= 16777216 \log. 2 \\ &= 16777216 \times \cdot 30103 \\ &= 5050445\cdot 33248; \end{aligned}$$

hence, since the index or characteristic of this logarithm is 5050445, the number answering to it must consist of 5050446 figures, so that the number x, if printed, would fill nine volumes of 350 pages each, allowing 40 lines to a page, and 40 figures to a line.

ON EXPONENTIAL EQUATIONS.

(152.) An exponential equation is an equation in which the unknown term is expressed in the form of a power with an unknown index; thus, the following are exponential equations:

$$a^x = b,\ x^x = a,\ ab^x = c,\ \&c.$$

(153.) When the exponential is of the form a^x, the value of x is readily found by logarithms; for of $a^x = b$, we have

$$x \log. a = \log. b,\ \therefore x = \frac{\log. b}{\log. a}.$$

Also, if $ab^x = c$, put $b^x = y$, then $a^y = c$, whence $y \log. a = \log. c$;

$$\therefore y = \frac{\log. c}{\log. a},\ \text{put this } = d,\ \text{then } b^x = d,\ \therefore\ x = \frac{\log. d}{\log. b}.$$

(154.) But if the equation be of the form $x^x = a$, then the value of x may be obtained by the following rule of double position.

Find by trial two numbers as near the true value of x as possible,

and substitute them separately for x, then, as the difference of the results is to the difference of the two assumed numbers, so is the difference of the true result, and either of the former, to the difference of the true number and the supposed one belonging to the result last used; this difference therefore being added to the supposed number, or subtracted from it, according as it is too little or too great, will give the true value nearly.

And if this near value be substituted for x, as also the nearest of the first assumed numbers, unless a number still nearer be found, and the above operations be repeated, we shall obtain a still nearer value of x, and in this way we may continually approximate to the true value of x.

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

$$\text{Here } x \log. x = \log. 100 \doteq 2,$$

and upon trial x is found to lie between 3 and 4;

$\therefore$ substituting each of these, we have

$$3 \log. 3 = 1{\cdot}4313639$$
$$\text{and } 4 \log. 4 = 2{\cdot}4082400$$

$$\therefore \quad {\cdot}9768761 = \text{difference of results.}$$

$$\therefore 9768761 : 1 :: 4082400 : {\cdot}418,$$

whence $4 - {\cdot}418 = 3{\cdot}582 = x$ nearly.

Now this value is found, upon trial, to be rather too small; and 3·6 is found to be rather too great; therefore, substituting each of these, we have

$$3{\cdot}582 \log. 3{\cdot}582 = 1.9848779$$
$$3{\cdot}6 \log. 3.6 \quad = 2.0026890$$

$$\therefore \quad {\cdot}0178111 = \text{diff. of results.}$$

$$\therefore 0178111 : {\cdot}018 :: 002689 : 002717,$$

whence $3{\cdot}6 - {\cdot}002717 = 3{\cdot}597283 = x$ very nearly.

The operation of solving the equation $x^x = a$ may be conducted differently, by using logarithms throughout; thus, in the equation

$x \log. x = \log. a$, call $\log. x$, x'; and $\log. a$, a'; then $xx' = a'$, $\therefore$ $\log. x + \log. x' = \log. a'$, that is, $x' + \log. x' = \log. a'$; hence we have to find a number x', which, when increased by its log., shall be equal to $\log. a'$, which may be effected by the rule of position before given.

Thus, taking the same example as before, viz. $x^x = 100$, we have $\log. 100 = 2 = a'$, and $\log. 2 = \cdot 3010300$; $\therefore$ $x' + \log. x' = \cdot 3010300$, and the nearest value of x' in the tables below the true value is $\cdot 55597$, which added to its log. $\bar{1}\cdot 7450514$, gives $\cdot 3010214$, and $\therefore$ the nearest value above the truth is $\cdot 55598$, which added to its log. $\bar{1}\cdot 740592$, gives $\cdot 3010392$; hence, by the rule:

$$\begin{array}{rr} 3010392 & 3010300 \\ 3010214 & 3010214 \\ \hline 178 & 86 \\ \hline \end{array}$$

$$\therefore\ 178 : 1 :: 86 : 483,$$

consequently $x' = \cdot 55597483 = \log. x$, $\therefore$ $x = 3\cdot 597284$.

If a be less than unity this solution fails, since a' is then negative, and therefore the $\log. a'$ is unassignable. But if we put $x = \frac{1}{y}$, and $a = \frac{1}{b}$, we shall have, by substitution, the equation $b^y = y$, $\therefore$ $y \log. b = \log. y$; put $\log. b = b'$, and $\log. y = y'$, then $yb' = y'$, $\therefore$ $\log. y + \log. b' = \log. y'$, or $y' + \log. b' = \log. y'$; whence y' may be found by the preceding rule.

2. Given $x^x = 5$, to find an approximate value of x.

Ans. $x = 2\cdot 1289$.

3. Given $x^x = 2000$, to find an approximate value of x.

Ans. $x = 4\cdot 8278$.

COMPOUND INTEREST.

(155.) Interest is a certain sum paid for the use of money for any stated period, and when the interest of this money is regularly received, the money, or principal, is said to be at simple interest;

but when, instead of being regularly received, it is allowed to go to the increase of the principal, then the interest of the whole is called compound interest.

(156.) An annuity is a yearly income, or pension.

(157.) The present value of an annuity is that sum which, if put out at compound interest, shall amount to sufficient to pay the annuity at the time it becomes due.

(158.) PROBLEM 1. To find the amount of a given sum in any number of years at compound interest.

Let r represent the interest of $1l.$ for 1 year, and put $1l. + r = \text{R} =$ the amount in 1 year.

Then $1l. : \text{R} :: \text{R} : \text{R}^2 =$ the amount in 2 years,

$1l. : \text{R} :: \text{R}^2 : \text{R}^3 =$ the amount in 3 years,

&c.

Therefore R^n is the amount of $1l.$ in n years, and, consequently, the amount of £p is $p\text{R}^n$, $\therefore$ calling the amount a, we have $\log. a = \log. p + n \log. \text{R}$, and $\log. p = \log. a - n \log. \text{R}$.

$$Cor. 1. \text{ Log. R} = \frac{\log. a - \log. p}{n}, \text{ and } n = \frac{\log. a - \log. p}{\log. \text{R}}.$$

Therefore any one of the quantities a, p, R, n, may be found from having the others given.

Cor. 2. If $a = mp$, then

$$n = \frac{\log. mp - \log. p}{\log. \text{R}} = \frac{\log. m + \log. p - \log. p}{\log. \text{R}} = \frac{\log. m}{\log. \text{R}}.$$

(159.) If the interest, instead of being due yearly, is supposed to become due half-yearly, quarterly, or after any other given period, then n, of course, instead of representing years, represents some number of those periods, r being the interest for one period.

EXAMPLES.

1. How much would $300l.$ amount to in 4 years, at 4 per cent. per annum compound interest?

Here $p = 300$, $\text{R} = 1 + \frac{4}{100} = 1.04$, and $n = 4$;

$\therefore \log. a = \log. p + n \log. \text{R} = \log. 300 + 4 \log. 1{\cdot}04 = 2{\cdot}5452545.$

the number answering to which in the tables is 350·9575 ∴ the amount is 350*l*. 19*s*. 1¾*d*.

2. How much money must be placed out at compound interest to amount to 1000*l*. in 20 years, the interest being 5 per cent.?

$$\text{Here } a = 1000,\ \text{R} = 1 + \tfrac{5}{100} = 1{\cdot}05, \text{ and } n = 20;$$

$$\therefore \log. p = \log. a - n \log. \text{R} = \log. 1000 - 20 \log. 1{\cdot}05 = 2{\cdot}576214,$$

the number answering to which is 376.89:

∴ the principal is 376*l*. 17*s*. 9½*d*.

3. At what interest must 300*l*. be placed out to amount to 350*l*. 19*s*. 2*d*. in 4 years?

$$\text{Here } p = 300,\ a = 350{\cdot}957, \text{ and } n = 4;$$

$$\therefore \log. \text{R} = \frac{\log. a - \log. p}{n} = \frac{\log. 350{\cdot}957 - \log. 300}{n} = {\cdot}0170333,$$

the number answering to which is 1·04;

$$\therefore r = {\cdot}04, \text{ and } {\cdot}04 \times 100 = 4, \text{ the rate per cent.}$$

4. In how many years will 400*l*. amount to 540*l*. at 4 per cent. compound interest?

$$\text{Here } p = 400,\ a = 540, \text{ and } \text{R} = 1 + \tfrac{4}{100} = 1{\cdot}04:$$

$$\therefore n = \frac{\log. a - \log. p}{\log. \text{R}} = \frac{\log. 540 - \log. 400}{\log. 1{\cdot}04} = \frac{{\cdot}1303338}{{\cdot}0170333} = 7{\cdot}65 \text{ years.}$$

5. What will 600*l*. amount to in 6 years at 4½ per cent. compound interest, supposing the interest to be receivable half-yearly?

$$\text{Here } p = 600,\ n = 12, \text{ and } \text{R} = 1 + \frac{2{\cdot}25}{100} = 1{\cdot}0225;$$

$$\therefore \log. a = \log. p + n \log. \text{R} = \log. 600 + 12 \log. 1{\cdot}0225 = 2{\cdot}8941109;$$

the number answering to which is 783·63:

∴ the amount is 783*l*. 12*s*. 7*d*.

6. In what time will a sum of money double itself at 5 per cent. compound interest?

$$\text{Here } m = 2, \text{ and } \text{R} = 1{\cdot}05;$$

$$\therefore n = \frac{\log. m}{\log. \text{R}} = \frac{\log. 2}{\log. 1{\cdot}05} = \frac{{\cdot}3010300}{{\cdot}0211893} = 14{\cdot}206 = 14\tfrac{1}{5} \text{ years nearly.}$$

7. In what time will 500*l.* amount to 900*l.* at 5 per cent. compound interest?

Ans. in 12·04 years.

8. What would 200*l.* amount to, if placed out for 7 years at 4 per cent. compound interest?

Ans. 263*l.* 3*s.* 8½*d.*

9. At what rate of compound interest must 376*l.* 17*s.* 9*d.* be placed out to amount to 1000*l.* in 20 years?

Ans. 5 per cent.

10. In what time will a sum of money double itself at 3½ per cent. compound interest?

Ans. 20·149 years.

PROBLEM II. To find the amount when the principal is increased not only by the interest, but also by some other sum at the same time.

The amount of the original principal p in n years is $p\text{R}^n$, and if A be the sum that is continually added, the first A will be at interest $n-1$ years; the second will be at interest $n-2$ years, &c., and therefore the sum of their amounts is

$$\text{AR}^{n-1} + \text{AR}^{n-2} + \ldots\ldots \text{AR}^{n-n}, \text{ or}$$
$$\text{A}(\text{R}^{n-1} + \text{R}^{n-2} + \ldots\ldots 1).$$

Now the terms within the parenthesis form a geometrical progression, whose first term is R^{n-1}, and ratio R, therefore the sum will be $\text{A} \times \frac{\text{R}^n - 1}{\text{R} - 1}$; ∴ the whole amount is $p\text{R}^n + \text{A} \times \frac{\text{R}^n - 1}{r}$, or, when $\text{A} = p$, then $a = \frac{p(\text{R}^{n+1} - 1)}{r}$.

If, however, A is not added the nth year, then we have $a = p\text{R}^n + \frac{\text{AR}(\text{R}^{n-1} - 1)}{r}$, or when $\text{A} = p$, $a = \frac{p\text{R}(\text{R}^n - 1)}{r}$.

Cor. 1. If, instead of $p = \text{A}$, we have $p = 0$, then $a = \frac{\text{A}(\text{R}^n - 1)}{r}$; which expresses the amount of an annuity A, at compound interest left unpaid for n years.

Cor. 2. If P be the present value of the annuity A for n years, P must be such, that if it were put out at compound interest for n years, it would amount to the same sum as the annuity, that is, we must have $PR^n = \dfrac{A(R^n - 1)}{r}$, whence $P = \dfrac{A(1 - \frac{1}{R^n})}{r}$.

Cor. 3. If n be infinite, then $\dfrac{1}{R^n}$ will vanish, in which case we shall have $P = \dfrac{A}{r}$.

EXAMPLES.

1. Suppose 300*l.* be put out at compound interest, and that to the stock is yearly added 20*l.*, what will be the amount at the expiration of 6 years, the interest being 4 per cent.?

$$\text{Here } p = 300,\ A = 20, \text{ and } r = \cdot 04,$$

$$\therefore a = pR^n + \frac{AR(R^{n-1} - 1)}{r} = 300(1\cdot 04)^6 + \frac{20 \times 1\cdot 04\,[(1\cdot 04)^5 - 1]}{\cdot 04}.$$

Now log. $300(1\cdot 04)^6 = \text{log. } 300 + 6 \text{ log. } 1\cdot 04 = 2.5793211$
$= \text{log. } 379\cdot 595$,

and log. $(1\cdot 04)^5 = 5 \text{ log. } 1\cdot 04 = \cdot 0851665 = \text{log. } 1\cdot 216652$:
$\therefore a = 379\cdot 595 + 500 \times 1\cdot 04 \times \cdot 216652 = 492.254 =$ 492*l.* 5*s.* 1*d.*

2. How much will an annuity of 50*l.* amount to in 20 years at $3\frac{1}{2}$ per cent. compound interest?

$$\text{Here } A = 50,\ r = \frac{3.5}{100} = \cdot 035, \text{ and } n = 20,$$

$$\therefore a = \frac{A(R^n - 1)}{r} = \frac{50(1\cdot 035^{20} - 1)}{\cdot 035};$$

now log. $(1\cdot 035)^{20} = 20 \text{ log. } 1\cdot 035 = \cdot 298806 = \text{log. } 1.989784$:

$$\therefore a = \frac{50 \times \cdot 989784}{.035} = 1413l.\ 19s.\ 7d.$$

3. Required the present value of an annuity of 50*l.* which is to continue 20 years at $3\frac{1}{2}$ per cent. compound interest.

By the last question, the amount is 1413*l*. 19*s*. 7*d*., also $R = 1{\cdot}035$, and $n = 20$:

$\therefore PR^n = 141398$, $\therefore \log. P = \log. 1413.98 - n \log. R = 2{\cdot}8516372$
$= \log. 710{\cdot}62 = 710l.\ 12s.\ 4d.$

4. If the annual rent of a freehold estate be £A, what is its present value at 5 per cent. compound interest?

Here, since n is infinite, $P = \frac{A}{r} = \frac{A}{\cdot 05} = 20A$; that is, the present value is 20 years' purchase.

5. What is the amount of an annuity of 30*l*. forborne 16 years at 4½ per cent. compound interest?

Ans. 681*l*. 11*s*. 4½*d*.

6. In what time will an annuity of 20*l*. amount to 1000*l*. at 4 per cent. compound interest?

Ans. 28 years.

7. What is the present value of a perpetual annuity of £A, allowing 3 per cent. compound interest?

Ans. 33⅓A.

(160.) We shall conclude this chapter on the application of logarithms with the following problem.

8. Suppose the interest of £1 for the xth part of a year to be $\frac{r}{x}$, it is required to determine the amount of £a when x is infinitely great.

Calling the amount A, we have

$$A = A\left(1 + \frac{r}{x}\right)^x$$

and taking the Napierian logarithms of each side of this equation,

$$\begin{aligned} \log. A &= \log. a + x \log. \left\{1 + \frac{r}{x}\right\} \\ &= \log. a + x\left\{\frac{r}{x} - \frac{r^2}{2x^2} + \frac{r^3}{3x^3} - \&c.\right\} \\ &= \log. a + r - \frac{r^2}{2x} + \frac{r^3}{3x^2} - \&c. \end{aligned}$$

Let now x be infinitely great, then the terms having x in the denominators vanish, so that

$$\log. A = \log. a + r.$$

Put $\log. n$ for r, then

$$\log. A = \log. a + \log. n = \log. an.$$

$$\therefore A = an;$$

that is, the amount is equal to a times the number whose Napierian logarithm is r.—(See Note B at the end.)

CHAPTER VII.

ON SERIES.

THE DIFFERENTIAL METHOD.

(161.) THE DIFFERENTIAL METHOD is the method of finding the successive differences of the terms of a series, and thence any intermediate term, or the sum of the whole series.

PROBLEM I.

(162.) To find the first term of any order of differences.

Let a, b, c, d, e, &c. represent any series; then, if the successive differences of the terms be taken, these differences will form a new series, which is called the first order of differences; in like manner, if the successive differences of the terms of this last series be taken, a new series, called the second order of differences, will be obtained, &c. Thus,

1st order of differences,

$$
\begin{array}{lllll}
b-a, \quad c-b, & d-c, & e-d, & \&c. \\
\quad\quad\quad\; b-a & \quad c-b & \quad d-c & \\
\hline
\text{2d order, } c-2b+a, & d-2c+b, & e-2d+c, & \&c. \\
 & \quad c-2b+a & \quad d-2c+b & \\
\hline
\text{3d order} \ldots\ldots\ldots\ldots & d-3c+3b-a, & e-3d+3c-b, & \&c. \\
\quad \&c. & & \quad \&c. &
\end{array}
$$

Now, since in the first order the first term in any difference is the same, except the sign, as the second in the succeeding difference, in subtracting any difference from the succeeding, the first term in the former must be placed under the second term of the latter, and, consequently, the same must take place in every succeeding order.

Hence the coefficients of the several terms, composing either of the differences belonging to any order, are respectively the same as the coefficients of the terms in the expanded binomial, being generated exactly in the same way,* the terms that are subtracted being in reality added with contrary signs.

Therefore, representing the first difference of the 1st, 2d, 3d, &c. order respectively by $\Delta^1, \Delta^2, \Delta^3$, &c. we have for the first difference of the nth order,

---

* Thus,

$$
\begin{array}{ll}
1-1 & = \text{coefficients of the first order,} \\
1-1 & \\
\hline
1-1 & \\
\;\;-1+1 & \\
\hline
1-2+1 & = \text{coefficients of the second order,} \\
1-1 & \\
\hline
1-2+1 & \\
\;\;-1+2-1 & \\
\hline
1-3+3-1 & \ldots\ldots\ldots\ldots\ldots\ldots \text{ third order,} \\
\quad \&c. & \quad\quad\quad\quad \&c.
\end{array}
$$

When n *is an even number,*

$$\Delta^n = a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2 \,.\, 3}d + \&c.$$

When n *is an odd number,*

$$\Delta^n = -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2 \,.\, 3}d - \&c.$$

EXAMPLES.

Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

Here a, b, c, d, e, &c. = 1, 8, 27, 64, 125, &c. and $n = 4$.

$$\therefore\ a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2 \,.\, 3}d +$$

$$\frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4}e = a - 4b + 6c - 4d + e = 1 - 32$$
$$+ 162 - 256 + 125 = 0;$$

hence the first term of the fourth order is 0.

2. Required the first term of the fifth order of differences of the series 1, 3, 3^2, 3^3, 3^4, &c.

Here a, b, c, d, e, &c. = 1, 3, 9, 27, 81, &c. and $n = 5$,

$$\therefore\ -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2 \,.\, 3}d -$$

$$\frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4}e + \&c. = -a + 5b - 10c + 10d -$$

$5e + f = -1 + 15 - 90 + 270 - 405 + 243 = 32 =$ the first term of the fifth order of differences.

3. Required the first term of the third order of differences of the series 1, 2^3, 3^3, 4^3, &c.

Ans. 6.

4. Required the first term of the fourth order of differences of the series 1, 6, 20, 50, 105, &c.

Ans. 2.

PROBLEM II.

(163.) To find the nth term of the series a, b, c, d, e, &c.

Let $\Delta^1, \Delta^2, \Delta^3, \Delta^4$, &c. represent the first term in the first, second, third, fourth, &c. order of differences respectively, then, in the general expressions for the first term of the nth order, we shall have, by making n successively equal to 1, 2, 3, &c., and transposing,

$$\begin{aligned}
b &= \quad a + \Delta^1, \\
c &= -a + 2b + \Delta^2, \\
d &= \quad a - 3b + 3c - \Delta^3, \\
e &= -a + 4b - 6c + 4d + \Delta^4, \\
f &= \quad a - 5b + 10c - 10d + 5e + \Delta^5, \\
\&c. &= \qquad \&c.
\end{aligned}$$

Or, by substitution,

$$\begin{aligned}
b &= a + \Delta^1, \\
c &= a + 2\Delta^1 + \Delta^2, \\
d &= a + 3\Delta^1 + 3\Delta^2 + \Delta^3, \\
e &= a + 4\Delta^1 + 6\Delta^2 + 4\Delta^3 + \Delta^4, \\
f &= a + 5\Delta^1 + 10\Delta^2 + 10\Delta^3 + 5\Delta^4 + \Delta^5 \\
\&c. &= \qquad \&c.
\end{aligned}$$

where the coefficients of $a, \Delta^1, \Delta^2, \Delta^3$, &c. in the $n+1$th term of the series a, b, c, d, &c. are the same as the coefficients of the terms of a binomial raised to the nth power, that is, the $n+1$th term is

$$a + n\Delta^1 + \frac{n(n-1)}{2}\Delta^2 + \frac{n(n-1)(n-2)}{2\,.\,3}\Delta^3 + \&c.$$

and therefore the nth term is

$$a+(n-1)\Delta^1+\frac{(n-1)(n-2)}{2}\Delta^2+\frac{(n-1)(n-2)(n-3)}{2\,.\,3}\Delta^3+\&c.$$

EXAMPLES.

1. Required the tenth term of the series 1, 4, 8, 13, 19, &c.

1, 4, 8, 13, 19,
3, 4, 5, 6,
1, 1, 1,
0.

Here the first terms of the differences are 3, 1, and 0;

that is, $\Delta^1 = 3$, $\Delta^2 = 1$, and $\Delta^3 = 0$, also $a = 1$, and $n = 10$;

$$\therefore a + (n-1)\Delta^1 + \frac{(n-1)(n-2)}{2}\Delta^2 = 1 + 27 + 36 = 64,$$

which is the tenth term required.

2. Required the twelfth term in the series, $1^3, 2^3, 3^3, 4^3, 5^3$, &c.

$$\begin{array}{c} 1,\ 8,\ 27,\ 64,\ 125, \\ 7,\ 19,\ 37,\ 61, \\ 12,\ 18,\ 24, \\ 6,\ 6, \\ 0. \end{array}$$

Here $\Delta^1=7$, $\Delta^2=12$, $\Delta^3=6$, $\Delta^4=0$, also $a=1$, and $n=12$;

$$a + (n-1)\Delta^1 + \frac{(n-1)(n-2)}{2}\Delta^2 + \frac{(n-1)(n-2)(n-3)}{2 \cdot 3}\Delta^3$$

$= 1 + 77 + 660 + 990 = 1728$, the twelfth term.

3. Given the logarithms of the numbers 101, 102, 104, and 105, to find the logarithm of 103.

Here, of five consecutive terms, four are given to find the intermediate one. To accomplish this with perfect accuracy would require us to know the value of Δ^4, which is itself not generally determinable without the term sought. But the four logarithms which are here given are themselves not strictly accurate, being indeed carried only to a limited number of decimals, usually seven; and, from the slow increase of the logarithms at the part of the table where these occur, we may easily assure ourselves that Δ^4 can have no significant figure in the first seven places of decimals: it may therefore be rejected, without introducing error. Hence, regarding Δ^4 as 0, we have, for the determination of the term c sought, the equation

$$e = -a + 4b - 6c + 4d$$

$$\therefore c = \frac{4(b+d) - (a+e)}{6}$$

which expression is thus calculated,

$$
\begin{aligned}
a &= \log.\ 101 = 2.0043214 \\
b &= \log.\ 102 = 2.0086002 \\
d &= \log.\ 104 = 2.0170333 \\
e &= \log.\ 105 = 2.0211893 \\
\therefore\ 4(b+d) &= 16.1025340 \\
(a+e) &= \ \ 4.0255107 \\
& \quad 6\,)\,12.0770233 \\
c = \log.\ 103 &= \ \ 2.0128372
\end{aligned}
$$

In this manner may any intermediate term in a series be calculated, provided always that, p being the number of given terms, the difference Δ^p may be rejected, without committing sensible error. The student who wishes for further information upon this subject of *interpolation*, more especially in reference to its utility in computing logarithms, may consult Chap. ii. of the Essay on the Computation of Logarithms before referred to.

4. Required the twentieth term of the series 1, 3, 5, 7, &c.

Ans. 39.

5. Required the twentieth term of the series 1, 3, 6, 10, 15, &c.

Ans. 210.

6. Required the fifteenth term of the series $1, 2^2, 3^2, 4^2$, &c.

Ans. 225.

7. Given the logarithms of 50, 51, 52, 54, and 55, to find the logarithm of 53.

Ans. log. 53 = 1.7242759.

PROBLEM III.

(164.) To find the sum of n terms of a series.

Let the proposed series be as before a, b, c, d, &c.; then, by means of the general expressions in last Problem, we shall be able to find the sum of n terms of this series, provided we can devise another series, such that either the $n + 1$th or the nth term may always be equal to n terms of the proposed. Now the series whose

$n+1$th term equals the sum of n terms of the proposed at once presents itself; it is the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

of which the first differences, viz.

$$a, \quad b, \quad c, \quad d, \quad \&c.$$

form the original series, and consequently that which is Δ^1 in the new series is the first term in the proposed, and that which is Δ^2 in the former is Δ^1 in the latter, and so on. Hence, referring to the general expression in last Problem, we have for the $n+1$th term of the new series, that is, for the sum of n terms of the proposed, the formula

$$S = na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 + \&c.$$

EXAMPLES.

1. Required the sum of n terms of the series 1, 3, 5, 7, &c.

$$\begin{array}{c} 1,\ 3,\ 5,\ 7, \\ 2,\ 2,\ 2, \\ 0,\ 0. \end{array}$$

Here $\Delta^1 = 2$, and $\Delta^2 = 0$, also $a = 1$;

$$\therefore\ na + \frac{n(n-1)}{2}\Delta^1 = n^2 = \text{sum of } n \text{ terms.}$$

2. Required the sum of n terms of the series $1,\ 2^2,\ 3^2,\ 4^2,\ 5^2$, &c.

$$\begin{array}{c} 1,\ 4,\ 9,\ 16,\ 25, \\ 3,\ 5,\ 7,\ 9, \\ 2,\ 2,\ 2, \\ 0,\ 0. \end{array}$$

Here $\Delta^1 = 3$, $\Delta^2 = 2$, and $\Delta^3 = 0$, also $a = 1$;

$$\therefore\ na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2.3}\Delta^2 = \frac{2n+3n^2-3n}{2} + \frac{n^3-3n^2+2n}{3} = \frac{n(n+1)(2n+1)}{6} = \text{sum of } n \text{ terms.}$$

3. Required the sum of n terms of the series 1, 2, 3, 4, 5, &c.

Ans. $\dfrac{n^2+n}{2}$.

4. Required the sum of twelve terms of the series 1, 4, 8, 13, 19, &c.

Ans. 430.

5. Required the sum of n terms of the series 1, 3, 6, 10, 15, &c.

Ans. $\frac{n(n+1)(n+2)}{6}$.

6. Required the sum of n terms of the series 1, 2^3, 3^3, 4^3, &c.

Ans. $\frac{n^2(n+1)^2}{4}$.

7. Required the sum of n terms of the series, 1, 2^4, 3^4, 4^4, &c.

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

ON THE SUMMATION OF INFINITE SERIES.

(165.) An Infinite Series is a progression of quantities proceeding onwards without termination, but usually according to some regular law discoverable from a few of the leading terms.

(166.) A converging series is a series whose successive terms decrease or become less and less, as the series

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\text{\&c.}$$

x being any whole number. The *finite quantity* to which we continually approach, by summing up more and more of the leading terms, is the quantity to which the series converges, and to which it actually attains only when taken in all its infinitude of terms. Should the series be infinite in value, as well as in extent, it is not regarded as convergent, even though its terms successively diminish. The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c., for instance, is not considered to be convergent, as it does not tend to any limit, its value being infinite. (See the "Essay on Logarithms.")

(167.) A diverging series is one whose successive terms increase or become greater and greater; such is the series

$$\frac{1}{1+2}=1-2+4-8+16-\text{\&c.}$$

(168.) A neutral series is one whose terms are all equal, but have signs alternately + and —, as the series

$$\frac{1}{1+1} = 1 - 1 + 1 - 1 + 1 - \&c.$$

(169.) An ascending series is one in which the powers of the unknown quantity ascend, as in the series

$$a + bx + cx^2 + dx^3 + \&c.$$

(170.) A descending series is one in which the powers of the unknown quantity descend, as in the series

$$a + bx^{-1} + cx^{-2} + dx^{-3} + \&c.$$

(171.) The summation of series is the finding a finite expression equivalent to the series.

(172.) As different series are often governed by very different laws, the methods of finding the sum which are applicable to one class of series will not apply universally; a great variety of useful series may be summed by help of the following considerations:

(173.) I. Since $\frac{q}{n} - \frac{q}{n+p} = \frac{pq}{n(n+p)}$, $\therefore \frac{q}{n(n+p)} =$

$$\frac{1}{p}\left\{\frac{q}{n} - \frac{q}{n+p}\right\};$$

that is, any fraction of the form $\frac{q}{n(n+p)}$ is equal to $\frac{1}{p}$th the difference between the two fractions $\frac{q}{n}$ and $\frac{q}{n+p}$; hence, if this difference be known, the value of $\frac{q}{n(n+p)}$ will be known, whether $\frac{q}{n}$ and $\frac{q}{n+p}$ be known or not; and it therefore follows, that if there be any series of fractions, each having the form $\frac{q}{n(n+p)}$, the sum of the series will be equal to $\frac{1}{p}$th the difference between a series of fractions of the form $\frac{q}{n}$, and another of the form $\frac{q}{n+p}$, and, if this difference can be obtained, the sum of the proposed series may be readily found, whatever be the values of p, q, and n.

EXAMPLES.

1. Required the sum of the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +$ &c. continued to infinity.

Here $q = 1$, and $p = 1$, also $n = 1, 2, 3,$ &c. successively;

$$\therefore \left\{ \begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c. ad inf.} \\ -(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c. ad inf.}) \end{array} \right\} = 1 = \text{sum}.$$

2. Required the sum of the above series to n terms.

$$\left\{ \begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n} \\ -(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n} + \frac{1}{n+1}) \end{array} \right\} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

3. Required the sum of the series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} +$ &c. ad infinitum.

Here $p = 2$,

$$\left\{ \begin{array}{l} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{\&c. ad inf.} \\ -(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{\&c. ad inf.}) \end{array} \right\} = 1, \therefore \frac{1}{p} = \frac{1}{2} = \text{sum}.$$

4. Required the sum of the above series to n terms;

$$\left\{ \begin{array}{l} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots \frac{1}{2n-1} \\ -(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots \frac{1}{2n-1} + \frac{1}{2n+1}) \end{array} \right\} = 1 - \frac{1}{2n+1}$$

$$= \frac{2n}{2n+1}, \text{ and } \frac{1}{p}\text{th of this is } \frac{n}{2n+1} = \text{sum}.$$

5. Required the sum of the series $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7} +$ &c. to infinity.

Here $p = 3$,

$$\left\{ \begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{\&c.} \\ -(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{\&c.} \end{array} \right\} = 1 + \frac{1}{2} + \frac{1}{3} = 1\tfrac{5}{6},$$

and $\frac{1}{p}$th of this is $\frac{11}{18}$ = sum.

6. Required the sum of n terms of the above series,

$$\left.\begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots \frac{1}{n} \\ -(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots\ldots \frac{1}{n+3}) \end{array}\right\} = 1 + \frac{1}{2} + \frac{1}{3} -$$

$$\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right) = \frac{n}{n+1} + \frac{n}{2n+4} + \frac{n}{3n+9},$$

$$\therefore \frac{n}{3n+3} + \frac{n}{6n+12} + \frac{n}{9n+27} = \text{sum.}$$

7. Required the sum of the series $\frac{2}{3.5} - \frac{3}{5.7} + \frac{4}{7.9} - \frac{5}{9.11} + \&c.$

Here $p = 2$, and $q = 2, 3, 4,$ &c. successively;

$$\left.\begin{array}{l} \frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \&c. \\ -(\frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \&c.) \end{array}\right\} = \frac{2}{3} - 1 + 1 - 1 + 1 - 1 + \&c. =$$

$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, and $\frac{1}{p}$ of this sum is $\frac{1}{12} =$ sum.

8. Required the sum of the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \&c.$ ad infinitum.

This series is evidently the same as the following, viz.

$$1 + \frac{1}{3} + \frac{1}{2.3} + \frac{1}{2.5} + \&c.$$

and dividing by 2, it becomes

$$\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \&c.$$

whose sum is 1 (Ex. 1st); $\therefore$ the sum of the proposed series is 2.

9. Required the sum of the series $\frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \&c.$ ad infinitum.

This series is the same as $\frac{1}{4}(\frac{1}{3.2} + \frac{1}{6.3} + \frac{1}{9.4} + \&c.)$

$$= \frac{1}{12}(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} = \&c.) = (\text{ex. } 1)\, \frac{1}{12};$$

also the sum to n terms is $\frac{n}{12\,(n+1)}$

10. Required the sum of the series $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} +$ &c. ad infinitum.

Ans. $\frac{3}{4}$.

11. Required the sum of the series $\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} -$ &c. ad infinitum.

Ans. $\frac{1}{4}$.

12. Required the sum of the series $\frac{2}{3.5} - \frac{3}{5.7} + \frac{4}{7.9} - \frac{5}{9.11} +$ &c. ad infinitum.

Ans. $\frac{1}{12}$.

13. Required the sum of the last series to n terms.

Ans. $\frac{1}{12} \pm \frac{1}{4(2n+3)}$, according as n is odd or even.

14. Required the sum of the series $\frac{4}{1.5} + \frac{4}{5.9} + \frac{4}{9.13} + \frac{4}{13.17}$ + &c. ad infinitum.

Ans. 1.

(174.) 2. Also, since

$$\frac{q}{n(n+p)} - \frac{q}{(n+p)(n+2p)} = \frac{2pq}{n(n+p)(n+2p)}, \therefore \frac{q}{n(n+p)(n+2p)}$$
$$= \frac{1}{2p}\left\{\frac{q}{n(n+p)} - \frac{q}{(n+p)(n+2p)}\right\};$$

hence the sum of any series of fractions, each of which is of the form $\frac{q}{n(n+p)(n+2p)}$, is equal to $\frac{1}{2p}$, the difference between one series, whose terms are of the form $\frac{q}{n(n+p)}$, and another, whose terms are of the form $\frac{q}{(n+p)(n+2p)}$.

EXAMPLES.

1. Required the sum of the series $\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} +$ &c. ad infinitum.

Here $p = 1$, and $q = 4, 5, 6$, successively;

$$\therefore \left\{ \begin{array}{l} \frac{4}{1.2} + \frac{5}{2.3} + \frac{6}{3.4} + \&c. \\ \quad -(\frac{4}{2.3} + \frac{5}{3.4} + \&c.) \end{array} \right\} =$$

$\frac{4}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c.$ (art. 13, ex. 1) $= 2\frac{1}{2}$, and $\frac{1}{2p}$ of this is $1\frac{1}{4}$ $=$ sum.

2. Required the sum of $\frac{3}{5.8.11} + \frac{9}{8.11.14} + \frac{15}{11.14.17} + \&c.$ ad infinitum.

Here $p = 3$,

$$\left\{ \begin{array}{l} \frac{3}{5.8} + \frac{9}{8.11} + \frac{15}{11.14} + \&c. \\ \quad -(\frac{3}{8.11} + \frac{9}{11.14} + \&c.) \end{array} \right\} =$$

$$\frac{3}{5.8} + \frac{6}{8.11} + \frac{6}{11.14} + \&c. =$$

$\frac{3}{5.8} + \frac{1}{3} \left\{ \begin{array}{l} \frac{6}{8} + \frac{6}{11} + \frac{6}{14} + \&c. \\ -(\frac{6}{11} + \frac{6}{14} + \&c.) \end{array} \right\} = \frac{3}{5.8} + \frac{2}{8} = \frac{13}{40}$, and $\frac{1}{2p}$ of this is $\frac{13}{240} =$ sum.

3. Required the sum of the series $\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \&c.$ ad infinitum.

Ans. $\frac{1}{8}$.

4. Required the sum of the series $\frac{1}{1.3.5} + \frac{4}{3.5.7} + \frac{7}{5.7.9} + \frac{10}{7.9.11} + \&c.$ ad infinitum.

Ans. $\frac{5}{24}$.

5. Required the sum of the series

$$\frac{a}{n(n+p)(n+2p)} + \frac{a+b}{(n+p)(n+2p)(n+3p)} +$$

$$\frac{a+2b}{(n+2p)(n+3p)(n+4p)} + \&c. \text{ ad infinitum.}$$

Ans. $\frac{pa + bn}{2p^2n(n+p)}$

(175.) 3. Likewise, since

$$\frac{q}{n(n+p)(n+2p)} - \frac{q}{(n+p)(n+2p)(n+3p)} =$$

$$\frac{3pq}{n(n+p)(n+2p)(n+3p)}, \therefore \frac{q}{n(n+p)(n+2p)(n+3p)} =$$

$$\frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)} - \frac{q}{(n+p)(n+2p)(n+3p)}\right\};$$

therefore, any series of fractions, of the form

$\frac{q}{n(n+p)(n+2p)(n+3p)}$ is equal to $\frac{1}{3p}$, the difference between a

series of the form $\frac{q}{n(n+p)(n+2p)}$, and another of the form

$$\frac{q}{(n+p)(n+2p)(n+3p)}.$$

EXAMPLES.

1. Required the sum of the series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} +$ &c. ad infinitum.

Here $p = 1$,

$$\left.\begin{array}{l} \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \text{\&c.} \\ \quad -(\frac{1}{2.3.4} + \frac{1}{3.4.5} + \text{\&c.}) \end{array}\right\} = \frac{1}{1.2.3} = \tfrac{1}{6};$$

$$\therefore \frac{1}{3p}(\tfrac{1}{6}) = \tfrac{1}{18} = \text{sum}.$$

2. Required the sum of the series $\frac{1}{1.3.5.7} + \frac{2}{3.5.7.9} + \frac{3}{5.7.9.11} +$ &c. ad infinitum.

Here $p = 2$,

$$\left.\begin{array}{l} \frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \text{\&c.} \\ \quad -(\frac{1}{3.5.7} + \frac{2}{5.7.9} + \text{\&c.}) \end{array}\right\} =$$

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\text{\&c.}=\tfrac{1}{12};$$

$$\therefore \frac{1}{3p}(\tfrac{1}{12})=\tfrac{1}{72}=\text{sum}.$$

3. Required the sum of the series $\dfrac{2}{3.6.9.12}+\dfrac{5}{6.9.12.15}+\dfrac{8}{9.12.15.18}$ + &c. ad infinitum.

Ans. $\frac{21}{1701}$.

4. Required the sum of the series $\dfrac{6^2}{1.2.3.4}+\dfrac{7^2}{2.3.4.5}+\dfrac{8^2}{3.4.5.6}+$ &c. ad infinitum.

Ans. $\frac{89}{36}$.

(176.) In a similar manner, it may be shown that the sum of any series of fractions of the form

$$\frac{q}{n(n+p)(n+2p)\ldots\ldots(n+mp)}$$

is equal to $\dfrac{1}{mp}$ the difference between a series of the form

$$\frac{q}{n(n+p)(n+2p)\ldots\ldots[n+(m-1)p]},$$

and another of the form

$$\frac{q}{(n+p)(n+2p)\ldots\ldots(n+mp)}.$$

(177.) Again, since

$$\frac{a(a+b)(a+2b)\ldots.(a+pb)}{n(n+b)\ldots.[n+(p-1)b]}-\frac{a(a+b)(a+2b)\ldots.[a+(p+1)b]}{n(n+b)\ldots.(n+pb)}$$

$$=\frac{a(n-a-b)(a+b)(a+2b\ldots.(a+pb)}{n(n+b)(n+2b)\ldots.(n+pb)},$$

$$\therefore \frac{a(a+b)(a+2b)\ldots.(a+pb)}{n(n+b)(n+2b)\ldots.(n+pb)}=$$

$$\frac{1}{n-a-b}\left\{\frac{a(a+b)\ldots.(a+pb)}{n(n+b)\ldots.[n+(p-1)b]}-\frac{a(a+b)\ldots.[a+(p+1)b]}{n(n+b)\ldots.(n+pb)}\right\}.$$

Hence, any series of fractions of the form

$$\frac{a(a+b)\ldots\ldots(a+pb)}{n(n+b)\ldots\ldots(n+pb)}$$

is equal to $\frac{1}{n-a-b}$ the difference of a series of the form

$$\frac{a(a+b)\ldots\ldots(a+pb)}{n(n+b)\ldots\ldots[n+(p-1]b)}$$

and another of the form

$$\frac{a(a+b)\ldots\ldots[a+(p+1)b]}{n(n+b)\ldots\ldots(n+pb)}$$

EXAMPLES.

1. Required the sum of the series $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\frac{1.3.5.7}{2.4.6.8}+$ &c. to r terms.

Here $a=1$, $b=2$, and $n=2$,

$$\left\{\begin{array}{l} 1+\frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots\frac{1.3.5.7\ldots\ldots(2r-1)}{2.4.6\ldots\ldots(2r-2)} \\ -\left(\frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots\ldots\ldots\ldots\ldots\frac{1.3.5.7\ldots\ldots(2r+1)}{2.4.6\ldots\ldots2r}\right)\end{array}\right\}=$$

$$1-\frac{1.3.5.7\ldots\ldots(2r+1)}{2.4.6\ldots\ldots2r},$$

and $\frac{1}{n-a-b}$ of this is $\frac{1.3.5.7\ldots\ldots(2r+1)}{2.4.6\ldots\ldots2r}-1=$ sum of r terms when r is infinite, this expression is evidently infinite also.

2. Required the sum of the series

$$\frac{a}{n}+\frac{a(a+b)}{n(n+b)}+\frac{a(a+b)(a+2b)}{n(n+b)(n+2b)}+\text{ \&c. to } r \text{ terms,}$$

$$\left\{\begin{array}{l} a+\frac{a(a+b)}{n}+\ldots\frac{a(a+b)\ldots\ldots[a+(r-1)b]}{n(n+b)\ldots\ldots[n+(r-2)b]} \\ -\left(\frac{a(a+b)}{n}+\ldots\ldots\ldots\ldots\frac{a(a+b)\ldots\ldots(a+rb)}{n(n+b)\ldots\ldots[n+(r-1)b]}\right)\end{array}\right\}$$

$$=a-\frac{a(a+b)(a+2b)\ldots\ldots(a+rb)}{n.(n+b)\ldots\ldots[n+(r-1)b]};$$

$$\therefore \text{ sum} = \frac{a}{n-a-b} - \frac{a(a+b)(a+2b)\dots.(a+rb)}{(n-a-b)n(n+b)\dots.[n+(r-1)b]}.$$

If r be infinite, then this expression for the sum will become definite only in particular cases. Thus, if $n = a + 2b$, the second fraction in the above expression will be

$$\frac{a(a+b)}{b[a(r+1)b]},$$

which evidently vanishes when r is infinite, in which case the sum is $\frac{a}{n-a-b}$; the same fraction would, of course, vanish if n were greater than $a + 2b$. So that in these cases we should always have for the sum the definite result $\frac{a}{n-a-b}$.

But if n were equal to $a + b$, then the said fraction would become

$$\frac{a(a+b)(a+2b)\dots.(a+rb)}{0(a+b)(a+2b)\dots.(a+rb)} = \frac{a}{0}$$

and the sum would become $\frac{a-a}{0} = \frac{0}{0}$, an expression of no definite signification in its present form. The sum presents itself under the same indefinite form even when r is finite, provided $n = a + b$, as will appear by inspecting the general expression.

3. Required the sum of r terms of the series $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \text{\&c.}$

Ans. $\frac{2.4.6.8\dots.(2r+2)}{3.5.7.9\dots.(2r+1)} - 2.$

4. Required the sum of the series $\frac{2}{5.6} + \frac{2.3}{5.6.7} + \frac{2.3.4}{5.6.7.8} + \text{\&c.}$ ad infinitum.

Ans. $\frac{2}{15}$.

(178.) As every summable infinite series may be supposed to arise from the expansion of some fractional expression, the value of the series may often be obtained by first assuming it equal to a fraction whose denominator is such, that when the series is multiplied by it,

the product may be finite, which product being equal to the numerator of the assumed fraction, determines its value, as in the examples following.*

EXAMPLES.

1. Required the sum of the infinite series $x + x^2 + x^3 +$ &c.

Assume the series equal to $\frac{z}{1-x}$;

$$\begin{array}{rl}
\text{then,} & x + x^2 + x^3 + \&c. \\
 & 1 - x \\ \hline
 & x + x^2 + x^3 + \&c. \\
 & \quad - x^2 - x^3 - \&c. \\ \hline
z = & x \quad * \quad * \quad * \\ \hline
\end{array}$$

that is, $x + x^2 + x^3 + \&c. = \frac{x}{1-x}$.

If $x = \frac{1}{2}$, then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$.

If $x = \frac{1}{3}$, then $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \&c. = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

&c. &c.

2. Required the sum of the infinite series

$$x - x^2 + x^3 - x^4 + \&c.$$

Assume the series equal to $\frac{z}{1+x}$;

$$\begin{array}{rl}
\text{then,} & x - x^2 + x^3 - x^4 + \&c. \\
 & 1 + x \\ \hline
 & x - x^2 + x^3 - x^4 + \&c. \\
 & \quad x^2 - x^3 + x^4 - \&c. \\ \hline
z = & x \quad * \quad * \quad * \\ \hline
\end{array}$$

* This method, however, is very limited in its application, on account of the difficulty of determining a suitable denominator for the assumed fraction. But a direct and easy method of summing every infinite series of which the generating function is rational, will be found in the chapter on the "Theory of Equations."

that is, $x - x^2 + x^3 - x^4 + \text{\&c.} = \dfrac{x}{1+x}$.

If $x = \frac{1}{2}$, then $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \text{\&c.} = \dfrac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$.

If $x = 1$, then $1 - 1 + 1 - 1 + 1 - \text{\&c.} = \dfrac{1}{1+1} = \frac{1}{2}$.

If $x = 2$, then $2 - 4 + 8 - 16 + \text{\&c.} = \dfrac{2}{1+2} = \frac{2}{3}$.

&c. &c. &c.

3. Required the sum of the infinite series $x + 2x^2 + 3x^3 + \text{\&c.}$

Assume the series $= \dfrac{z}{(1-x)^2} = \dfrac{z}{1 - 2x + x^2}$;

then,

$$
\begin{array}{rl}
 & x + 2x^2 + 3x^3 + \text{\&c.} \\
 & 1 - 2x + x^2 \\
\hline
 & x + 2x^2 + 3x^3 + \text{\&c.} \\
 & \quad - 2x^2 - 4x^3 - \text{\&c.} \\
 & \qquad\qquad x^3 + \text{\&c.} \\
\hline
z = & x \quad * \quad * \quad * \\
\hline
\end{array}
$$

that is, $x + 2x^3 + 3x^3 + \text{\&c.} = \dfrac{x}{(1-x)^2}$.

If $x = \frac{1}{2}$, then $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \text{\&c.} = \dfrac{\frac{1}{2}}{\frac{1}{4}} = 2$.

If $x = \frac{1}{3}$, then $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \text{\&c.} = \dfrac{\frac{1}{3}}{(\frac{2}{3})^2} = \frac{3}{4}$.

&c. &c. &c.

4. Required the sum of the infinite series $x + 4x^2 + 9x^3 + 16x^4 + \text{\&c.}$

Assume the sum $= \dfrac{z}{(1-x)^3}$;

then $(1-x)^3 \times (x + 4x^2 + 9x^3 + \text{\&c.}) = x + x^2$;

$\therefore x + 4x^2 + 9x^3 + \text{\&c.} = \dfrac{x + x^2}{(1-x)^3} = \dfrac{x(1+x)}{(1-x)^3}$.

If $x = \frac{1}{2}$, then $\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \text{\&c.} = 6$.

&c. &c.

PROMISCUOUS EXAMPLES.

1. Required the sum of the series $\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \frac{5}{3.4.2^3} +$ &c. ad infinitum.

$$\left.\begin{array}{l} \frac{3}{1.2} + \frac{4}{2.2^2} + \frac{5}{3.2^3} + \&c. \\ \quad -(\frac{3}{2.2} + \frac{4}{3.2^2} + \&c.) \end{array}\right\} = \frac{3}{1.2} - (\frac{2}{2.2^2} + \frac{3}{3.2^3} + \&c.)$$

$$= \frac{3}{1.2} - (\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \&c.) = \tfrac{3}{2} - \tfrac{1}{2} = 1 = \text{sum}.$$

2. Required the sum of the series $\frac{5}{1.2.3.2^2} + \frac{6}{2.3.4.2^3} + \frac{7}{3.4.5.2^4}$ + &c. ad infinitum.

Ans. $\frac{1}{4}$.

3. Required the sum of the series $x + 3x^2 + 6x^3 + 10x^4 +$ &c. ad infinitum.

Ans. $\frac{x}{(1-x)^3}$.

4. Required the sum of n terms of the series $\frac{1}{4.8} - \frac{1}{6.10} + \frac{1}{8.12} -$ &c.

Ans. $\frac{n}{16(1+n)} - \frac{n}{12(3+2n)}$.

5. Required the sum of the series $\frac{1}{8.18} + \frac{1}{10.21} + \frac{1}{12.24} + \frac{1}{14.27}$ + &c. ad infinitum.

Ans. $\frac{3}{80}$.

10. Required the sum of the series $\frac{10.18}{2.4.9.12} + \frac{12.21}{4.6.12.15} + \frac{14.24}{6.8.15.18}$ + &c. ad infinitum.

Ans. $\frac{19}{54}$.

ON RECURRING SERIES.

(179.) A recurring series is one, each of whose terms, after a certain number, bears a uniform relation to the same number of those which immediately precede.

(180.) It is obvious that a variety of infinite series will arise from developing different fractional expressions: those however which generate recurring series are always of a particular form.

(181.) The fraction $\frac{a}{a' + b'x}$, for instance, is of this kind, for the series which arises from the actual division is recurring thus:

$$
\begin{array}{l}
a' + b'x\,)\,a \quad \left(\frac{a}{a'} - \frac{ab'x}{a'^2} + \frac{ab'^2x^2}{a'^3} - \&c. \right. \\
\qquad a + \frac{ab'x}{a'} \\
\qquad \overline{\qquad - \frac{ab'x}{a'}} \\
\qquad - \frac{ab'x}{a'} - \frac{ab'^2x^2}{a'^2} \\
\qquad \overline{\qquad\qquad \frac{ab'^2x^2}{a'^2}} \\
\qquad\qquad \frac{ab'^2x^2}{a'^2} + \frac{ab'^3x^3}{a'^3} \\
\qquad\qquad \overline{\qquad - \frac{ab'^3x^3}{a'^3}} \\
\qquad\qquad\qquad \&c.
\end{array}
$$

where it is obvious that each term, commencing at the second, is equal to that which immediately precedes multiplied by $-\frac{b'x}{a'}$, which quantity is called the *scale of relation* of the terms, or $\frac{b'}{a'}$ is the scale of relation of the coefficients; therefore, representing the terms of the series by A, B, C, D, &c., we have

$$\text{A} = \frac{a}{a'}, \text{ whence } a'\text{A} - a = 0,$$

$$\text{B} = -\frac{b'x}{a'}\text{A} \ldots\ldots b'x\text{A} + a'\text{B} = 0,$$

$$\text{C} = -\frac{b'x}{a'}\text{B} \ldots\ldots b'x\text{B} + a'\text{C} = 0,$$

$$\text{D} = -\frac{b'x}{a'}\text{C} \ldots\ldots b'x\text{C} + a'\text{D} = 0,$$

&c. &c.

here we may observe, that the coefficients of A, B; of B, C; of C, D, &c., are the terms of the denominator of the generating fraction taken in reverse order.

(182.) The fraction $\dfrac{a + bx}{a' + b'x + c'x^2}$ is another of this kind; for if this be developed as that above, and similar substitutions be made, there will be found to result

$$\text{A} = \frac{a}{a'}, \text{ whence } \ldots\ldots a'\text{A} - a \ldots\ldots\ldots = 0,$$

$$\text{B} = \frac{b - b'\text{A}}{a'} \ldots\ldots\ldots b'\text{A} + a'\text{B} - b \ldots\ldots = 0,$$

$$\text{C} = -\frac{c'x^2\text{A} + b'x\text{B}}{a'} \ldots\ldots c'x^2\text{A} + b'x\text{B} + a'\text{C} = 0,$$

$$\text{D} = -\frac{c'x^2b + b'x\text{C}}{a'} \ldots\ldots c'x^2\text{B} + b'x\text{C} + a'\text{D} = 0,$$

&c. &c.

where each term, commencing at the third, is equal to the two immediately preceding multiplied respectively by $-\dfrac{c'x^2}{a'}$, $-\dfrac{b'x}{a'}$, which is therefore the scale of relation of the terms; also, the coefficients of A, B, C; of B, C, D, &c., are the terms of the generating fraction taken in reverse order.

(183.) The fraction $\dfrac{a + bx + cx^2}{a' + b'x + c'x^2 + d'x^3}$ is also one of the same kind, as its development will show; the scale of relation of the terms,

in the resulting series, being $-\frac{d'x^3}{a'}, -\frac{c'x^2}{a'}, -\frac{b'x}{a'}$, commencing at the fourth term. And, in general, the development of any rational fraction of the form

$$\frac{a + bx + cx^2 + \dots px^m}{a' + b'x + c'x^2 + \dots q'x^{m+1}}$$

will be a recurring series, in which any term, commencing at the $m + 2$th, will be equal to the $m + 1$ preceding multiplied by $-\frac{q'x^{m+1}}{a'}, -\frac{p'x^m}{a'}, \dots -\frac{c'x^2}{a'}, -\frac{b'x}{a'}$, respectively, which is therefore the scale of relation of the terms.

If $a' = 1$, then the scale of relation is $-q'x^{m+1}, -p'x^m, \dots\dots -c'x^2, -b'x$, being the several terms of the denominator taken in reverse order, the first term 1 being omitted.

PROBLEM I.

To find the sum of an infinite recurring series.

Let $A + B + C + D + \dots\dots + K + L + M + N$ represent a recurring series, and let it be supposed such, that each term, commencing at the fourth, depends upon the three preceding; then, as in Art. (182), we shall have, by supposing the terms in the generating fraction to be p, q, r, s, the following equations, viz.

$$\begin{aligned} sA + rB + qC + pD &= 0, \\ sB + rC + qD + pE &= 0, \\ sC + rD + qE + pF &= 0, \\ sD + rE + qF + pG &= 0, \\ \dots\dots\dots\dots & \\ sK + rL + qM + pN &= 0, \end{aligned}$$

and taking the sum of these equations, we have

$$s(A + B + C + D + \dots K) + r(B + C + D + E + \dots L) \\ + q(C + D + E + F + \dots M) + p(D + E + F + G + \dots N) = 0;$$

which, by putting S for the sum, becomes the same as

$$s(S - L - M - N) + r(S - A - M - N) + q(S - A - B - N) \\ + p(S - A - B - C) = 0;$$

from which equation we get s =

$$\frac{p(\text{A}+\text{B}+\text{C})+q(\text{A}+\text{B}+\text{N})+r(\text{A}+\text{M}+\text{N})+s(\text{L}+\text{M}+\text{N})}{p+q+r+s};$$

so that the sum may be determined from having the three first, and three last terms, with the scale of relation given; but if the series be infinite, and decreasing, the three last terms will vanish, and the sum will be

$$\frac{p(\text{A}+\text{B}+\text{C})+q(\text{A}+\text{B})+r\text{A}}{p+q+r+s}=\frac{\text{A}(p+q+r)+\text{B}(p+q)+\text{C}p}{p+q+r+s}$$

EXAMPLES.

1. Required the sum of the infinite recurring series

$$1+2x+8x^2+28x^3+100x^4+356x^5+\&c.$$

Here the scale of relation is $2x^2$, $3x$.

$\therefore$ the third term, $\text{C}=2x^2\text{A}+3x\text{B}$, whence

$$-2x^2\text{A}-3x\text{B}+\text{C}=0,$$

consequently, $s=-2x^2$, $r=-3x$, $q=1$, and $p=0$.

$$\therefore \text{ sum}=\frac{\text{A}(1-3x)+\text{B}}{1-3x-2x^2}=\frac{1-x}{1-3x-2x^2}.$$

2. Required the sum of the infinite recurring series

$$1+2x+3x^2+5x^3+8x^4+\&c.$$

the scale of relation being x^2, x.

Ans. $\dfrac{1+x}{1-x-x^2}$.

3. Required the sum of the infinite recurring series

$1+3x+5x^2+7x^3$, &c., the scale of relation being $-x^2$, $2x$.

Ans. $\dfrac{1+x}{1-2x+x^2}$.

4. Required the sum of the infinite recurring series

$$3+5x+7x^2+13x^3+23x^4+\&c.$$

the scale of relation being $-2x^3$, x^2, $2x$.

Ans. $\dfrac{3-x-6x^2}{1-2x-x^2+2x^3}$.

PROBLEM II.

To find the sum of any number of terms of a recurring series.*

This may be effected by means of the expression for S in the preceding problem, but more conveniently by subtracting from the sum of the series continued to infinity, the sum of all those terms which follow the nth; thus, if the nth term of the recurring series $A + B + C +$ &c. be T, then, putting S for the sum of all the terms to infinity, and S′ for the sum of those to infinity which follow T, we shall have, by last problem, $S - S' =$

$$\frac{A(p+q+r)+B(p+q)+Cp-U(p+q+r)-V(p+q)-Wp}{p+q+r+s}$$

$$= \frac{(A-U)(p+q+r)+(B-V)(p+q)+(C-W)p}{p+q+r+s} +$$

the sum of n terms.

EXAMPLES.

1. Required the sum of n terms of the series

$$1 + 2x + 3x^2 + \ldots nx^{n-1}.$$

Here the scale of relation is $-x^2, 2x$;

$$\therefore C = -x^2A + 2xB, \text{ whence } x^2A - 2xB + C = 0,$$

$$\therefore s = x^2, \ r = -2x, \ q = 1, \ p = 0, \text{ also } U = (n+1)x^n;$$

$$V = (n+2)x^{n+1}, \text{ and } W = (n+3)x^{n+2};$$

---

* The finding the sum of a finite number of terms of a recurring series supposes that the general term of the series is previously known: to discover the general term is, however, by far the most perplexing part of the problem, it being often attended with considerable difficulties. The only general way in which it can be discovered is derived from considering the generating fraction

$$\frac{a + bx + cx^2 + \ldots px^m}{a' + b'x + c'x^2 + \ldots q'x^{m+1}},$$

as the same as

$$(a + bx + cx^2 + \ldots px^m)(a' + b'x + c'x^2 + \ldots q'x^{m+1})^{-1},$$

which may be expanded, and the general term of the resulting series obtained, by the MULTINOMIAL THEOREM.

consequently, sum $= \dfrac{(\text{A} - \text{U})(1 - 2x) + \text{B} - \text{V}}{1 - 2x + x^2} =$

$$\frac{[1 - (n+1)x^n](1 - 2x) + 2x - (n+2)x^{n+1}}{1 - 2x + x^2} =$$

$$\frac{1 - (n+1)x^n + nx^{n+1}}{1 - 2x + x^2}$$

2. Required the sum of n terms of the series

$$1 + 3x + 5x^2 + 7x^3 + \&c.$$

the scale of relation being $-x^2$, $2x$.

Ans. $\dfrac{1 + x - (2n+1)x^n + (2n-1)x^{n+1}}{1 - 2x + x^2}$.

ON THE METHOD OF INDETERMINATE COEFFICIENTS.

(184.) The method of indeterminate coefficients, which is used to develop fractional and surd expressions, consists in assuming the proposed expression equal to a series with indeterminate, or unknown coefficients; and if this assumed series be multiplied by the denominator of its equivalent fraction, or raised to the power necessary to free from radicals its equivalent surd, then, by equating the coefficients of the homologous terms in the resulting equation, the several values of the assumed coefficients will become known.

EXAMPLES.

1. Required the development of $\dfrac{a}{a' + b'x}$ by the method of indeterminate coefficients.

Assume $\dfrac{a}{a' + b'x} = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 + \&c.$;

then multiplying each side by $a' + b'x$, and transposing, we have

$$\left.\begin{matrix}\text{A}a' \\ -a\end{matrix}\right. + \left.\begin{matrix}\text{B}a' \\ \text{A}b'\end{matrix}\right\} x \left.\begin{matrix}+\text{C}a' \\ +\text{B}b'\end{matrix}\right\} x^2 \left.\begin{matrix}+\text{D}a' \\ +\text{C}b'\end{matrix}\right\} x^3 + \&c. = 0:$$

$$\text{whence}\begin{cases} \text{A}a' - a = 0, \text{ therefore } \text{A} = \dfrac{a}{a'} \\ \text{B}a' + \text{A}b' = 0 \quad . \quad . \quad . \quad \text{B} = -\dfrac{b'}{a'}\text{A} \\ \text{C}a' + \text{B}b' = 0 \quad . \quad . \quad . \quad \text{C} = -\dfrac{b'}{a'}\text{B} \\ \text{D}a' + \text{C}b' = 0 \quad . \quad . \quad . \quad \text{D} = -\dfrac{b'}{a'}\text{C} \end{cases}$$
$$\&\text{c.} \qquad\qquad\qquad \&\text{c.}$$

$$\therefore \frac{a}{a' + b'x} = \frac{a}{a'} - \frac{b'}{a'}\text{A}x - \frac{b'}{a'}\text{B}x^2 - \frac{b'}{a'}, \text{C}x^3 - \&\text{c.}$$

the same as was before found from actual division.

2. Required the development of $\sqrt{a^2 + x^2}$ by this method.

Assume $\sqrt{a^2 + x^2} = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 + \&\text{c.}$

then, by squaring each side, and transposing, we have

$$\begin{matrix} \text{A}^2 \\ -a^2 \end{matrix} + 2\text{AB}x + \left.\begin{matrix} 2\text{AC} \\ +\text{B}^2 \\ -1 \end{matrix}\right\} x^2 \left.\begin{matrix} +2\text{AD} \\ +2\text{BC} \end{matrix}\right\} x^3 \left.\begin{matrix} +2\text{A} \\ +\text{C}^2 \end{matrix}\right\} x^4 + \&\text{c.} = 0;$$

$$\text{whence}\begin{cases} \text{A}^2 - a^2 = 0, \text{ therefore } \text{A} = a \\ 2\text{AB} = 0 \quad . \quad . \quad . \quad \text{B} = 0 \\ 2\text{AC} - 1 = 0 \quad . \quad . \quad . \quad \text{C} = \dfrac{1}{2a} \\ 2\text{AD} + 2\text{BC} = 0 \quad . \quad . \quad . \quad \text{D} = 0 \end{cases}$$
$$\&\text{c.} \qquad\qquad\qquad \&\text{c.}$$

$$\therefore \sqrt{a^2 + x^2} = a + \frac{x^2}{2a} - \frac{x^4}{8a^2} + \&\text{c.}$$

3. Required the development of $\dfrac{x}{1+x+x^2}$ by the same method

Here, since the first term of the series must contain x,

$$\text{assume } \frac{x}{1+x+x^2} = \text{A}x + \text{B}x^2 + \text{C}x^3 + \text{D}x^4 + \&\text{c.}$$

then we have

$$\left.\begin{matrix} \text{A} \\ -1 \end{matrix}\right\} x \left.\begin{matrix} +\text{A} \\ +\text{B} \end{matrix}\right\} x^2 \left.\begin{matrix} +\text{A} \\ +\text{B} \\ +\text{C} \end{matrix}\right\} x^3 \left.\begin{matrix} +\text{B} \\ +\text{C} \\ +\text{D} \end{matrix}\right\} x^4 \left.\begin{matrix} +\text{C} \\ +\text{D} \\ +\text{E} \end{matrix}\right\} x^5 + \&\text{c.} = 0;$$

$$\text{whence}\begin{cases} A-1 \quad\quad\quad = 0, \text{ therefore } A = \;\;\;1 \\ A+B \quad\quad\;\; = 0 \quad . \quad . \quad . \quad B = -1 \\ A+B+C = 0 \quad . \quad . \quad . \quad C = \;\;\;0 \\ B+C+D = 0 \quad . \quad . \quad . \quad D = \;\;\;1 \\ C+D+E = 0 \quad . \quad . \quad . \quad E = -1; \end{cases}$$

$$\therefore \frac{x}{1+x+x^2} = x - x^2 + x^4 - x^5 + x^7 - \&c.$$

4. Required the development of $\sqrt{1-x}$ by this method.

$$\text{Ans. } 1 - \frac{x}{2} - \frac{x^2}{2\cdot 4} - \frac{3x^3}{2\cdot 4\cdot 6} - \frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8} - \&c.$$

5. Required the development of $\frac{1+2x}{1-x-x^2}$ by this method.

$$\text{Ans. } 1 + 3x + 4x^2 + 7x^3 + 11x^4 + \&c.$$

6. Required the development of $\frac{1}{1-2ax+x^2}$ by the above method.

$$\text{Ans. } 1 + 2ax + (4a^2-1)x^2 + (8a^3-4a)x^3 + \&c.$$

ON THE MULTINOMIAL THEOREM.

(185.) THE MULTINOMIAL THEOREM is a formula which exhibits the general development of $(a+bx+cx^2+dx^3+\&c.)^{\frac{p}{q}}$ in a series ascending according to the power of x. It may be investigated as follows:

Assume

$$(a+bx+cx^2+\&c.)^{\frac{p}{q}} = A+Bx+Cx^2+\&c.$$

Similarly,

$$(a+by+cy^2+\&c.)^{\frac{p}{q}} = A+By+Cy^2+\&c.$$

Put for abridgment

$$(a+bx+cx^2+\&c.)^{\frac{1}{q}} = X, \quad (a+by+cy^2+\&c.)^{\frac{1}{q}} = Y;$$

then,

$$\frac{X^p - Y^p}{X^q - Y^q} = \frac{B(x-y) + C(x^2-y^2) + D(x^3-y^3) + \&c.}{b(x-y) + c(x^2-y^2) + d(x^3-y^3) + \&c.}$$

$$= \frac{B + C(x+y) + D)x^2 + xy + y^2) + \&c.}{b + c(x+y) + d(x^2 + xy + y^2) + \&c.}$$

Now when $x = y$ then $X = Y$, in which case we know (135) that the first side of the equation becomes

$$\frac{pX^{p-1}}{qX^{q-1}} = \frac{p}{q}\,.\,X^{p-q} = \frac{p}{q}(a + bx + cx^2 + \&c.)^{\frac{p}{q}-1};$$

and the second side becomes

$$\frac{B + 2Cx + 3Dx^2 + \&c.}{b + 2cx + 3dx^2 + \&c.}$$

Multiplying, therefore, each of these sides by

$$(a + bx + cx^2 + \&c.)\,(b + 2cx + 3dx^2 + \&c.)$$

and we have

$$\frac{p}{q}(a + bx + cx^2 + \&c.)^{\frac{p}{q}}\,(b + 2cx + 3dx^2 + \&c.)$$
$$= (a + bx + cx^2 + \&c.)\,(B + 2Cx + 3Dx^2 + \&c.)$$

or substituting for simplicity's sake n for $\frac{p}{q}$, and putting the assumed series for the second factor in the first member of this equation, we have

$$n(A + Bx + Cx^2 + \&c.)\,(b + 2cx + 3dx^2 + \&c.) =$$
$$(a + bx + cx^2 + \&c.)\,(B + 2Cx + 3Dx^2 + \&c.)$$

that is, by actually performing the multiplications here indicated,

$$\begin{array}{r|r|r|l} nAb + Bb & nx + cb & nx^2 + Db & nx^3 + \&c. \\ + 2Ac & + 2Bc & + 2Cc & \\ & + 3Ad & + 3Bd & \\ & & + 4Ae & \end{array}$$

is equal to

$$\begin{array}{r|r|r|l} Ba + 2Ca & x + 3Da & x^2 + 4Ea & x^3 + \&c. \\ + Bb & + 2Cb & + 3Db & \\ & + Bc & + 2Cc & \\ & & + Bd & \end{array}$$

Now A is obviously $= a^n$, therefore, by comparing the coefficients of the like terms in the above expressions, we shall have

$$\mathrm{B}a = n\mathrm{A}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{whence } \mathrm{B} = na^{n-1}b$$

$$2\mathrm{C}a = (n-1)\mathrm{B}b + 2n\mathrm{A}c \quad . \quad . \quad . \quad . \quad . \quad \mathrm{C} = \frac{(n-1)\mathrm{B}b + 2na^n c}{2a},$$

$$3\mathrm{D}a = (n-2)\mathrm{C}b + (2n-1)\mathrm{B}c + 3n\mathrm{A}d \quad . \quad . \quad . \quad \mathrm{D} = \frac{(n-2)\mathrm{C}b + (2n-1)\mathrm{B}c + 3na^n d}{3a},$$

$$4\mathrm{EA} = (n-3)\mathrm{D}b + (2n-2)\mathrm{C}c + (3n-1)\mathrm{B}d + 4n\mathrm{A}e \ldots \mathrm{E} = \frac{(n-3)\mathrm{D}b + (2n-2)\mathrm{C}c + (3n-1)\mathrm{B}d + 4na^n e}{4a},$$

&c. &c. &c.

Hence it appears that

$$(a + bx + cx^2 + \&c.)^n = a^n + na^{n-1}bx + \frac{(n-1)\mathrm{B}b + 2na^n c}{2a}x^2 + \frac{(n-2)\mathrm{C}b + (2n-1)\mathrm{B}c + 3na^n d}{3a}x^3 +$$

$$\frac{(n-3)\mathrm{D}b + (2n-2)\mathrm{C}c + (3n-1)\mathrm{B}d + 4na^n e}{4a}x^4 + \&c.$$

where B represents the coefficient of the second term, C that of the third, D that of the fourth, &c.

Or, if we put $na^{n-1} = \text{Q}$, this equation will become

$$(a + bx + cx^2 + \&c.)^n = a^n + \text{Q}bx + \left(\frac{(n-1)\text{B}b}{2a} + \text{Q}c\right)x^2 + \left(\frac{(n-2)\text{C}b + (2n-1)\text{B}c}{3a} + \text{Q}d\right)x^3 +$$

$$\left(\frac{(n-3)\text{D}b + (2n-2)\text{C}c + (3n-1)\text{B}d}{4a} + \text{Q}e\right)x^4 + \&c.$$

which is a very commodious form for practice.

EXAMPLES.

1. What is the cube of the series $1 + x + x^2 + x^3 + x^4 + \&c.$?

Here $a, b, c,$ &c. are each $= 1$, also $\text{Q} = 3$, therefore

$$a^n = 1 = \text{A},$$
$$\text{Q}b = 3 = \text{B},$$
$$\frac{(n-1)\text{B}b}{2a} + \text{Q}c = 6 = \text{C},$$
$$\frac{(n-2)\text{C}b + (2n-1)\text{B}c}{3a} + \text{Q}d = 10 = \text{D},$$
$$\frac{(n-3)\text{D}b + (2n-2)\text{C}c + (3n-1)\text{B}d}{4a} + \text{Q}e = 15 = \text{E},$$
&c.

$$\therefore (1 + x + x^2 + x^3 + \&c.)^3 = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \&c.$$

2. What is the square root of the series $1 + x + x^2 + x^3 + \&c.$?

Here a, b, c, &c. are each $= 1$, also $Q = \frac{1}{2}$, therefore

$$a^n = 1 = A,$$

$$Qb = \tfrac{1}{2} = B,$$

$$\frac{(n-1)Bb}{2a} + Qc = \tfrac{3}{8} = C,$$

$$\frac{(n-2)cb + (2n-1)Bc}{3a} + Qd = \tfrac{5}{16} = D,$$

$$\frac{(n-3)Db + (2n-2)Cc + (3n-1)Bd}{4a} + Qe = \tfrac{35}{128} = E;$$

&c.

$$\therefore (1 + x + x^2 + x^3 + \&c.)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 + \&c.$$

3. What is the cube of the series $2x + 3x^2 + 4x^3 + \&c.$?

Ans. $8x^3 + 36x^4 + 102x^5 + 231x^6 + \&c.$

4. What is the cube root of the series $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.$?

Ans. $1 + \frac{1}{6}x + \frac{1}{12}x^2 + \frac{25}{648}x^3 + \frac{383}{9720}x^4$ &c.

ON THE REVERSION OF SERIES.

(186.) To revert a series is to express the value of the unknown quantity in it by means of another series involving the powers of some other quantity.

1. Let the series be of the form $ax + bx^2 + cx^3 + \&c. = y$; then, in order to express the value of x in terms of y, assume $x = Ay + By^2 + Cy^3 + \&c.$, and substitute this value for x in the proposed series, which will, in consequence, become, when y is transposed,

$$\left.\begin{matrix} aA \\ -1 \end{matrix}\right\} y \left.\begin{matrix} + aB \\ + bA^2 \end{matrix}\right\} y^2 \left.\begin{matrix} + ac \\ + 2bAB \\ + cA^3 \end{matrix}\right\} y^3 \left.\begin{matrix} + qD \\ + 2bAC \\ + bB^2 \\ + 3cA^2B \\ + dA^4 \end{matrix}\right\} y^4 + \&c. = 0,$$

$$\text{whence}\left\{\begin{array}{ll} a\text{A}-1 \quad \ldots\ldots\ldots \quad =0, \therefore \text{A}= & \dfrac{1}{a}, \\ a\text{B}+b\text{A}^2 \quad \ldots\ldots\ldots \quad =0 \;..\; \text{B}= & -\dfrac{b}{a^3}, \\ a\text{C}+2b\text{AB}+c\text{A}^3 \ldots\ldots \quad =0 \;..\; \text{C}= & \dfrac{2b^2-ac}{a^5} \\ a\text{D}+2b\text{AC}+b\text{B}^2+3c\text{A}^2\text{B}+da^4=0 \;..\; \text{D}= & -\dfrac{5b^3-5abc+a^2d}{a^7}, \end{array}\right.$$

&c. &c.

consequently, $x=\dfrac{1}{a}y-\dfrac{b}{a^3}y^2+\dfrac{2b^2-ac}{a^5}y^3$

$$-\frac{5b^3-5abc+a^2d}{a^7}y^4+\&c.$$

2. If the series be of the form $ax+bx^3+cx^5+\&c.$, where the even powers of x are absent, then we shall have, instead of the above,

$$x=\frac{1}{a}y-\frac{b}{a^4}y^3+\frac{3b^2-ac}{a^7}y^5-\frac{12b^3+a^2d-8abc}{a^{10}}y^7+\&c.*$$

EXAMPLES.

1. Given the series $x+x^2+x^3+\&c.=y$, to express the value of x in terms of y.

Here a, b, c, &c. are each 1;

$$\text{therefore } \frac{1}{a}=1,$$

$$-\frac{b}{a^3}=-1,$$

$$\frac{2b^2-ac}{a^5}=1,$$

$$-\frac{5b^3-5abc+a^2d}{a^7}=-1,$$

&c. &c.

$$\therefore\; x=y-y^2+y^3-y^4+\&c.$$

* When the series is expressed by means of another, as

$$ax+bx^2+cx^3+\&c.=ay+\beta y^2+\gamma y^3+\&c.$$

the value of x is to be obtained exactly in the same way, by assuming $x=\text{A}y+\text{B}y^2+\text{C}y^3+\&c.$, and substituting this value in the place of x in the first series, as above.

2. It is required to revert the series

$$2x + 3x^3 + 4x^5 + 5x^7 + \&c. = y.$$

Here $a = 2$, $b = 3$, $c = 4$, &c.

$$\text{therefore } \frac{1}{a} = \tfrac{1}{2},$$

$$-\frac{b}{a^4} = -\tfrac{3}{16},$$

$$\frac{3b^2 - ac}{a^7} = \tfrac{19}{128},$$

$$-\frac{12b^3 + a^2b - 8abc}{a^{10}} = -\tfrac{152}{1024};$$

$$\therefore\ x = \tfrac{1}{2}y - \tfrac{3}{16}y^3 + \tfrac{19}{128}y^5 - \tfrac{152}{1024}y^7 + \&c.$$

3. Given the series $x - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \frac{1}{8}x^4 + \&c. = y$, to find the value of x in terms of y.

Ans. $x = y + \frac{1}{2}y^2 + \frac{1}{4}y^3 + \frac{1}{8}y^4 + \&c.$

4. Given the series $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c. = y$, to find the value of x in terms of y.

Ans. $x = y + \frac{1}{3}y^3 + \frac{2}{15}y^5 + \frac{17}{315}y^7 + \&c.$

5. Given the series $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2\,.\,3} + \dfrac{x^4}{2\,.\,3\,.\,4} + \&c. = y$, to find the value of x in terms of y.

Ans. $x = y - 1 - \dfrac{(y-1)^2}{2} + \dfrac{(y-1)^3}{3} - \&c.$*

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* We know, from what has been said of logarithms, that this value of x is $=$ log. y; but if $x =$ log. y $\therefore e^x = y$, consequently,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\,.\,3} + \&c.,$$

which is the exponential theorem otherwise established at (136).

CHAPTER VIII.

ON INDETERMINATE EQUATIONS.

(187.) EQUATIONS are said to be indeterminate, or unlimited, when they admit of an indefinite, or unlimited number of solutions, which they will always do when the number of unknown quantities exceeds the number of independent equations. The equation $ax - by = c$, for instance, is unlimited, for $x = \frac{c+by}{a}$, where y may be any value whatever, therefore x and y admit of an infinite number of values that will satisfy the equation $ax - by = c$; and such must evidently always be the case when one of the unknowns is expressible only by means of another unknown, each then admitting of an infinite number of values. The number of solutions in *integer* numbers is, however, often determinable. If, for instance, $ax + by = c$, then $x = \frac{c - by}{a}$, and, therefore, to have integer values of x and y, the question will be limited to the finding all the integer values of y that will make $\frac{c - by}{a}$ an integer. The limits of possibility, in equations of this kind, will be investigated in the following propositions. The symbol $<$ signifies *less than*, and $>$ signifies *greater than;* thus, $A < B$, means that A is *less* than B, and $A > B$, means that A is *greater* than B.

PROPOSITION I.

If a and b be any two numbers prime to each other, and if each of the terms

$$b,\ 2b,\ 3b,\ 4b,\ \ldots\ (a-1)b,$$

be divided by a, all the resulting remainders will be different.

For, if it be supposed that the remainders will not all be different, let any two of the above terms, as mb, nb, leave the same remainder r; then, representing the respective quotients by q, q', we must have

$$qa + r = mb,$$
$$\text{and } q'a + r = nb;$$

therefore, by subtraction, $a(q - q') = b(m - n)$, whence $\dfrac{b(m-n)}{a}$ is an integer number; but neither b, nor $m - n$, is divisible by a, the former being prime to it, and the latter less than a, since both m and n are less, therefore $\dfrac{b(m-n)}{a}$ cannot be an integer, and, consequently, the supposition cannot be admitted.

Cor. 1. Hence, since the remainders are all different, and are $a - 1$ in number, each being necessarily less than a, it follows that they include all numbers from 1 to $a - 1$.

Cor. 2. Therefore, since some one of the remainders will be 1, it follows that some number x less than a may be found that will make $bx - 1$ exactly divisible by a; or, which is the same thing, the equation $bx - ay = 1$ is always possible in integers, if a and b be prime to each other.

If, however, a and b be not prime to each other, the equation will be impossible in integers, for a and b having, in this case, a common measure, one side of the equation $bx - ay = 1$ would be divisible by it, and the other not.

Cor. 3. Since $bx - ay = 1$ is always possible, it follows, by changing the signs, that $ay - bx = -1$ is also possible; hence $ax - by = \pm 1$ is always possible in integers, if a and b be prime to each other.

PROPOSITION II.

If a and b be prime to each other, the equation

$$ax - by = \pm c$$

will admit of an infinite number of solutions in integer numbers.

For, since the equation $ax' - by' = \pm 1$ is possible, the equation

$$acx' - bcy' = \pm c$$

is possible, which, by putting x for cx', and y for cy', becomes

$$ax - by = \pm c,$$

being the same as the proposed equation.

Let now one solution be $x = p$, and $y = q$, then

$$ap - bq = ax - by, \text{ or } ax - ap = by - bq,$$

$$\therefore \frac{a(x-p)}{b(y-q)} = 1, \text{ and therefore } \frac{x-p}{y-q} = \frac{b}{a} = \frac{mb}{ma},$$

$$\text{or } x - p = mb, \text{ and } y - q = ma;$$

$$\therefore x = p + mb, \text{ and } y = q + ma;$$

and since m may be any value whatever, from 0 to infinity, the number of values of x and y may be infinite.

Cor. Since p and q are integers, and since m may be either positive or negative, m may be so assumed, that x shall be less than b, or that y shall be less than a, for making m equal to $0, -1, -2, -3$, &c. successively, we shall have

$$x = p,\ p - b,\ p - 2b, \text{ \&c. successively,}$$

$$\text{and } y = q,\ q - a,\ q - 2a, \text{ \&c. successively,}$$

where it is obvious that one of the values of x must be less than b, and one of the values of y less than a, whatever be the values of p and q.

PROPOSITION III.

The equation $ax + by = c$ is always possible in integers, if a and b be prime to each other, and if

$$c > (ab - a - b).$$

For let $c = (ab - a - b) + r$, then the equation becomes

$$ax + by = (ab - a - b) + r,$$

which is possible if

$$x = \frac{ab - a - b - by + r}{a} = b - 1 - \frac{(y+1)b - r}{a}$$

be an integer; but, since $b-1$ is an integer, the possibility depends upon

$$\frac{(y+1)b - r}{a} = x'$$

being an integer, or, putting $y + 1 = y'$, upon the possibility of the equation $by' - ax' = r$, which has been already established (Prop.

2); let then y' be less than a, or $y+1<a$ (Prop. 2, Cor.), then, in the equation

$$\frac{(y+1)\,b-r}{a}=x',$$

x' must be less than $b-1$, and it therefore follows that

$$x=b-1-\frac{(y+1)\,b-r}{a}=b-1-x'$$

must be some integer number; hence the equation $ax+by=c$ is always possible when a and b are prime to each other, and $c>(ab-a-b)$.

Schol. The two last propositions will evidently be useful in discovering the possibility or impossibility of equations of this kind, and also in enabling us to propose them with proper restrictions.

PROBLEM I.

To find the integer values of x and y in the equation

$$ax-by=c.$$

Since $x=\dfrac{by+c}{a}$ must be a whole number, it follows that if the division of $by+c$ by a be actually performed, the remainder $py+d$ must be divisible by a, that is, $\dfrac{py+d}{a}$ must represent a whole number; also, if from $\dfrac{ay}{a}$ the nearest multiple to it of $\dfrac{py+d}{a}$ be taken, the remainder, which may be represented by $\dfrac{qy+e}{a}$, must be a whole number, and q must be less than p; if again the difference of $\dfrac{py+d}{a}$, and the nearest multiple to it of $\dfrac{qy+e}{a}$ be taken, the remainder, which may be represented by $\dfrac{ry+f}{a}$, must also be a whole number, and r will be less than q; hence, by proceeding in this way, we shall at length arrive at a remainder of the form $\dfrac{y+k}{a}$, in which the coefficient of y is 1. Now the least value that can be

20 *

given to y, in order that this expression may be a positive whole number, will evidently, when k is negative, be equal to the remainder arising from the division of k by a; but, when k is positive, the least value will be equal to a *minus* this remainder: Hence, since the subtraction of fractions does not produce any change on the common denominators, the numerators only being operated upon, the process will be the same in effect by the following rule.

(167.) Having reduced the given equation to the form $x = \frac{by + c}{a}$, perform the division of $by + c$ by a, and call the remainder $py + d$. Take the difference of ay and the nearest multiple to it of $py + d$; then the difference of $py + d$ and nearest multiple to it of the remainder; then the difference of the preceding remainder and the nearest multiple to it of this last; and so on, till we get a remainder of the form $y - k$, or $y + k$, when the least value of y will, in the former case, be the remainder R, arising from dividing k by a, and in the latter case it will be a *minus* R.*

EXAMPLES.

1. Given $21x + 17y = 2000$, to find all the positive values of x and y in whole numbers.

Here $x = \frac{2000 - 17y}{21} = 95 - \frac{17y - 5}{21}$, and $a = 21$,

$$\begin{array}{l} 21y \quad\;\; = ay \\ 17y - 5 = py - d \\ \hline \;\; 4y + 5 \\ \qquad\;\; 4 \\ \hline 16y + 20 \\ \hline \;\; y - 25 = y - k; \\ \hline \end{array}$$

* This rule does not differ much from that given by Mr. Nicholson, in the Mathematical Companion, for 1819; it appears, however, to be rather more simple.

now $\frac{25}{21}$ gives a remainder $= 4 =$ the least value of y, which, substituted in the above expression for x, gives $\dfrac{2000 - 68}{21} = 92 =$ the greatest value of x, and by adding 21, the coefficient of x, continually to the least value of y, and subtracting 17, the coefficient of y, from the greatest value of x, we shall have all the possible values as follow:

$x = 92$	75	58	41	24	7
$y = 4$	25	46	67	88	109.

2. Given $19x = 14y - 11$, to find x and y in whole numbers.

Here $x = \dfrac{14y - 11}{19}$, $y = \dfrac{19x + 11}{14} = x + \dfrac{5x + 11}{14}$, and $a = 14$.

$$\begin{array}{r} 5x + 11 \\ 3 \\ \hline 15x + 33 \\ 14x \\ \hline x + 33 \\ \hline \end{array}$$

Now $\frac{33}{14}$ gives a remainder $= 5$, $\therefore 14 - 5 = 9 =$ the least value of x, and, since in this example the less x is the less will y be, we have, by substitution, $\dfrac{171 + 11}{14} = 13 =$ the least value of y, the number of solutions being indefinite.

3. Exhibit the number of different ways in which it is possible to pay 20*l.* in half-guineas and half-crowns only.

Let x represent the half-guineas, and y the half-crowns, then, by reducing to sixpences, we have

$$21x + 5y = 800;$$

$$\therefore x = \frac{800 - 5y}{21} = 38 - \frac{5y - 2}{21},$$

$$\begin{array}{r} 5y - 2 \\ 4 \\ \hline 20y - 8 \\ 21y \\ \hline y + 8 \\ \hline \end{array}$$

$\therefore$ R $= 8$, and $21 - 8 = 13 =$ the least value of y; and $\therefore$ 35 is the greatest value of x, consequently, if we add 21 continually to the least value of y, and subtract 5 from the greatest value of x, we shall have all the possible values; thus,

$x = 35$	30	25	20	15	10	5
$y = 13$	34	55	76	97	118	139

or the number of solutions, besides the one first obtained, might have been determined without this trouble, for the number of times that 5 can be continually subtracted from 35, so that the remainders may be all positive, is evidently one less than the quotient of 35 by 5, viz. 6: had this division left a remainder, the number of solutions would have been a unit more, that is, the whole quotient.

4. Given $5x + 11y = 254$, to find all the different values of x and y in positive whole numbers.

Ans.

$x = 9$	20	31	42
$y = 19$	14	9	4.

5. Given $11x + 35y = 500$, to find the least integer value of x.

Ans. 20.

6. Given $19x - 117y = 11$, to find the least integral values of x and y.

Ans. $x = 56$, and $y = 9$.

7. Is it possible to pay 50*l.* by means of guineas and three-shilling pieces only?

Ans. Impossible.

8. A person bought sheep and lambs for 8 guineas; the sheep cost 1*l.* 6*s.* a piece, and the lambs 15*s.* How many of each did he buy?

Ans. 3 sheep and 6 lambs.

9. Is the equation $7x + 13y = 71$ possible or impossible?

Ans. Impossible.

PROBLEM II.

To determine *à priori* the number of solutions that the equation

$$ax + by = c$$

will admit of.

Let such integral values of x' and y' be found, that we may have $ax' - by' = 1$, which we have shown to be always possible (Prop. 1, Cor. 2);

$$\text{then, } acx' - bcy' = c, \ \therefore \ ax + by = acx' - bcy',$$

and, consequently, we must have $x = cx' - mb$, and $y = ma - cy'$, where m may be any number taken at pleasure, that will make these values of x and y positive integers; but, if no such value of m can be found, it will be a proof that the proposed equation is impossible in positive integers, and, on the contrary, as many suitable values of m as can be found, so many solutions will the equation admit of, and no more. Hence, because we must have $cx' > mb$, and $cy' < ma$, the whole number of solutions will be expressed by the difference between the integral parts of

$$\frac{cx'}{b}, \text{ and } \frac{cy'}{a};$$

because, as m must be less than the first of these fractions, and greater than the second, the difference of their integral parts will evidently express the number of different values of m, except when $\frac{cx'}{b}$ is a complete integer; in which case, since $m < \frac{cx'}{b}$, the difference of the integral parts would be one more than the number of different values of m, therefore, when the expression $\frac{cx'}{b}$ is an integer, we must consider $\frac{b}{b}$ as a fraction, and reject it therefrom; but this must not be done with the other quantity $\frac{cy'}{a}$, because

$$m > \frac{cy'}{a}.$$

EXAMPLES.

1. Required the number of solutions that the equation $9x + 13y = 2000$ will admit of in positive integers.

In the equation $9x' - 13y' = 1$, we have

$$x' = \frac{13y' + 1}{9} = y' + \frac{4y' + 1}{9};$$

therefore,

$$\begin{array}{r} 4y' + 1 \\ 2 \\ \hline 8y' + 2 \\ 9y' \\ \hline y' - 2 \end{array}$$

$\therefore\ y' = 2$, and $x' = 3$, hence the number of solutions is the integral part of $\frac{2000 \times 3}{13}$ — the integral part of $\frac{2000 \times 2}{9}$, which is 17.

2. In how many different ways is it possible to pay 140*l.* by means of guineas and three-shilling pieces only?

Ans. The payment is impossible.

3. In how many different ways may 1000*l.* be paid in crowns and guineas?

Ans. 190.

PROBLEM III.

To find the integer values of x, y, z, in the equation

$$ax + by + cz = d.$$

Let c be the greatest coefficient in this equation, then, since the values of x and y cannot be less than 1, the value of z cannot be greater than

$$\frac{d - a - b}{c};$$

if, therefore, we ascertain this limit, and then proceed as in Prob. 1, we shall at length arrive at a remainder of the form $y \pm z \pm k$, where, if 1, 2, 3, &c. up to the limit, be successively substituted for z, all the values of x and y may be exhibited, as in Prob. 1.

EXAMPLES.

1. Given $3x + 5y + 7z = 100$, to exhibit all the different values of x, y, and z, in integers.*

* This example is the same as that given by Mr. Bonnycastle, at page 232, vol. 1, of his Algebra, where he finds the number of solutions to be 7, "which," says he, "are all the integer values of x, y, z, that can be obtained from the given equation:" from the above, however, it appears that 41 is the whole number of solutions.

Here z cannot be greater than $\dfrac{100-3-5}{7}=13\frac{1}{7}$;

and by proceeding as in Prob. 1,

$$x=\frac{100-5y-7z}{3}=33-y-2z-\frac{2y+z-1}{3};$$

$$\begin{array}{c} 3y \\ 2y+z-1 \\ \hline y-z+1 \\ \hline \end{array}$$

now, by taking $z=1$, y becomes $=0$, and $x=31$; but this answer is inadmissible, because $y=0$ is not an integer, but, by adding 3, the coefficient of x, to this value of y, and subtracting 5, the coefficient of y, from the value of x, we shall obtain another answer, and, by repeating this process continually, we shall obtain all the possible values of x and y, for this value of z; and in a similar manner are the values of x and y to be found when $z=2$, &c., when all the possible solutions will be found to be 41 in number, and to be as follow:

$$z=1\left\{\begin{array}{l|l|l|l|l|l|} y=\ 3 & 6 & 9 & 12 & 15 & 18 \\ x=26 & 21 & 16 & 11 & 6 & 1 \end{array}\right. \qquad z=7\left\{\begin{array}{l|l|l|} y=\ 3 & 6 & 9 \\ x=12 & 7 & 2 \end{array}\right.$$

$$z=2\left\{\begin{array}{l|l|l|l|l|l|} y=\ 1 & 4 & 7 & 10 & 13 & 16 \\ x=27 & 22 & 17 & 12 & 7 & 2 \end{array}\right. \qquad z=8\left\{\begin{array}{l|l|l|} y=\ 1 & 4 & 7 \\ x=13 & 8 & 3 \end{array}\right.$$

$$z=3\left\{\begin{array}{l|l|l|l|l|} y=\ 2 & 5 & 8 & 11 & 14 \\ x=23 & 18 & 13 & 8 & 3 \end{array}\right. \qquad z=9\left\{\begin{array}{l|l|} y=2 & 5 \\ x=9 & 4 \end{array}\right.$$

$$z=4\left\{\begin{array}{l|l|l|l|} y=\ 3 & 6 & 9 & 12 \\ x=19 & 14 & 9 & 4 \end{array}\right. \qquad z=10\left\{\begin{array}{l|} y=3 \\ x=5 \end{array}\right.$$

$$z=5\left\{\begin{array}{l|l|l|l|} y=\ 1 & 4 & 7 & 10 \\ x=20 & 15 & 10 & 5 \end{array}\right. \qquad z=11\left\{\begin{array}{l|l|} y=1 & 4 \\ x=6 & 1 \end{array}\right.$$

$$z=6\left\{\begin{array}{l|l|l|l|} y=\ 2 & 5 & 8 & 11 \\ x=16 & 11 & 6 & 1 \end{array}\right. \qquad z=12\left\{\begin{array}{l|} y=2 \\ x=2 \end{array}\right.$$

It is obvious, from the above, that when the solutions are very numerous, the process will become tedious; but there is seldom any necessity to exhibit all the solutions at length, as is done here, since the object of inquiry is not so much to find the solutions themselves,

as to determine, *à priori*, the number that the equation admits of; the method of doing which will be pointed out in the next Problem: we shall, therefore, merely add another example to this Problem, as an exercise for the student.

2. Given $17x + 19y + 21z = 400$, to exhibit all the different values of x, y, and z, in integers.

Ans.

$z =$	1	2	3	4	5	6	11	12	13	14
$y =$	11	9	7	5	3	1	8	6	4	2
$x =$	10	11	12	13	14	15	1	2	3	4

PROBLEM IV.

To determine the number of solutions that the equation

$$ax + by + cz = d$$

will admit of, two, at least, of the coefficients a, b, c, being prime to each other.*

By Prob. 2, the number of solutions that the equation $ax+by=c$ will admit of, is expressed by the integral parts of

$$\frac{cx'}{b} - \frac{cy'}{a}$$

* When this is not the case, the proposed equation must be transformed to another, that shall have two, at least, of its coefficients prime to each other. Thus, if the equation be

$$12x + 15y + 20z = 100001,$$

by transposing $20z$, and dividing by 3, we have

$$4x + 5y = 33334 - 7z + \frac{z-1}{3};$$

$\therefore \frac{z-1}{3}$ is an integer, which call u, then $z = 3u + 1$; whence, by substitution, the proposed equation becomes

$$12x + 15y + 20(3u + 1) = 100001,$$

which, by transposing the 20, becomes divisible by 3, and we then have

$$4x + 5y + 20u = 33327;$$

in which equation, x and y have, of course, the same values as in the one proposed, and therefore the number of solutions must be the same; but in this last one, value of u may be 0, because

$$z = 3u + 1.$$

x' and y' being determined from the equation $ax' - by' = 1$; therefore, in the equation $ax + by = d - cz$, if we make $z = 1, 2, 3, 4$, &c. successively, then the number of solutions

$$\text{in the equation}\left\{\begin{array}{lll} ax+by=d-c & \text{will be the integ. pts. of} & \dfrac{(d-c)x'}{b} - \dfrac{(d-c)y'}{a} \\ ax+by=d-2c & \ldots\ldots\ldots & \dfrac{(d-2c)x'}{b} - \dfrac{(d-2c)y'}{a} \\ ax+by=d-3c & \ldots\ldots\ldots & \dfrac{(d-3c)x'}{b} - \dfrac{(d-3c)y'}{a} \end{array}\right.$$

&c. &c. &c.

the sum of which will be the whole number of solutions that the equation admits of, that is, if we take the sum of the integral parts of the arithmetical series

$$\frac{(d-c)x'}{b} + \frac{(d-2c)x'}{b} + \frac{(d-3c)x'}{b} + \frac{(d-4c)x'}{b} + \text{\&c.};$$

as also of the arithmetical series

$$\frac{(d-c)y'}{a} + \frac{(d-2c)y'}{a} + \frac{(d-3c)y'}{a} + \frac{(d-4c)y'}{a} + \text{\&c.},$$

the difference of the two will be the whole number of integral solutions; now in each of these series the first and last terms, as also the number of terms, are known, for the general terms being

$$\frac{(d-cz)x'}{b}, \text{ and } \frac{(d-cz)y'}{a},$$

we shall have the extremes by taking the extreme limits of z, that is, $z = 1$, and $z < \dfrac{d-a-b}{c}$, which last value of z also expresses the number of terms in the series.

If, therefore, we find the sums of the two whole series, and then the sum of the fractional parts in each, by deducting these last sums, each from the corresponding whole sum, the sum of the integral parts of each series will be obtained.

In summing the fractional parts, there will be no necessity to go through the whole series, for, as the denominator in each is con-

stant, these fractions will necessarily recur in periods, and the number in each period can never exceed the denominator;* it will therefore only be necessary to find the sum of the fractions in one period, and to multiply this sum by the number of periods, in order to get the sum of all the fractions, observing, however, that when there is not an exact number of periods, the overplus fractions must be summed by themselves, which may be readily done, since they will be the same as the leading terms of the first period; it must also be remembered that $\frac{b}{b}$ is to be considered as a fraction in the first series, as in Prob. 2.

EXAMPLES.

1. Given the equation $5x + 7y + 11z = 224$, to find the number of solutions which it admits of in integers.

Here the greatest limit of $z < \frac{224 - 5 - 7}{11}$ is 19;

also in the equation $5x' - 7y' = 1$, we have $x' = 3$, and $y' = 2$, also $a = 5$, and $b = 7$;

therefore, the two series, of which the sums are required, beginning with the least terms, $\frac{(d - 19c)x'}{b}$, and $\frac{(d - 19c)y'}{a}$ will be

$$\frac{3.15}{7} + \frac{3.26}{7} + \frac{3.37}{7} + \ldots . \frac{3.113}{7},$$

and $$\frac{2.15}{5} + \frac{2.26}{5} + \frac{2.37}{5} + \ldots . \frac{2.113}{5};$$

the common difference in the first being $\frac{3.11}{7}$, and in the second

* This will appear from considering the above series; for, if in the first series d and c be prime to each other, and neither of them prime to b, each term will be wholly integral, that is, the fractions will all be 0. If b be prime to d, and not to c, the fractions will be all equal. If b be prime to c, but not to d, then the fractions will recur after the first integral term, which can never lie beyond the bth term; and, finally, if a, b, c, be all prime to each other, the series of fractions will always recur after the bth term. Similar observations evidently apply to the second series.

$\frac{2.11}{5}$, and the number of terms in each 19.

Now the sum of the first series is $928\frac{2}{7}$,

and the sum of the second .... $866\frac{2}{5}$;

also the first period of fractions, in the first series, is

$$\tfrac{3}{7} + \tfrac{1}{7} + \tfrac{6}{7} + \tfrac{4}{7} + \tfrac{2}{7} + \tfrac{7}{7} + \tfrac{5}{7} = 4,$$

and the first period, in the second series, is

$$0 + \tfrac{2}{5} + \tfrac{4}{5} + \tfrac{1}{5} + \tfrac{3}{5} = 2,$$

$\frac{7}{7}$ being considered as a fraction in the first period, but not $\frac{5}{5}$ in the second.

Hence the number of terms in each series being 19, we have two periods and five terms of the first series $= 2 \times 4 +$ the first five fractions $= 10\frac{2}{7}$, for the sum of all the fractions, and therefore $928\frac{2}{7} - 10\frac{2}{7} = 918 =$ sum of the integral terms of the first series: also in the second we have three periods and four terms $= 3 \times 2 + 1\frac{2}{5} = 7\frac{2}{5}$, and therefore $866\frac{2}{5} - 7\frac{2}{5} = 859 =$ sum of the integral terms of the second series; whence $918 - 859 = 59$ is the whole number of integral solutions.

In a similar manner may the number of solutions be obtained when there are four or more unknown quantities.

2. It is required to determine the number of integral solutions that the equation $3x + 5y + 7z = 100$ will admit of.

Ans. 41.

3. It is required to determine the number of integral solutions that the equation $7x + 9y + 23z = 9999$ will admit of.

Ans. 34365.

PROBLEM V.

To find the integral values of three unknown quantities in two equations.

When there are two equations and three unknown quantities, one of the unknowns may be exterminated as in simple equations (Art. 15, chap. 2), and the other unknowns may be found as in Prob. 1 of the present chapter.

EXAMPLES.

1. Given $\left\{\begin{array}{r} 2x + 5y + 3z = 51 \\ 10x + 3y + 2z = 120 \end{array}\right\}$, to find all the integral values of x, y, and z.

Here multiplying the first equation by 5, and subtracting the second, there results $22y + 13z = 135$;

$$\text{whence } z = \frac{135 - 22y}{13} = 10 - y - \frac{9y - 5}{13}.$$

$$\begin{array}{r} 13y \\ 9y - \ \ 5 \\ \hline 4y + \ \ 5 \\ 2 \\ \hline 8y + 10 \\ \hline y - 15 \\ \hline \end{array}$$

$\therefore y = 2$, and $z = 7$, which are the only values of y and z,

$$\therefore x = 10.$$

It should be remarked here that we are not to expect that when z and y admit of several values, each will satisfy the proposed equations; for the corresponding values of x may be fractional. All that we can infer is that the integral values of y and z, deduced as above, contain among them all those which can subsist with integral values of x; but what values do really so subsist can be ascertained only by trying each pair in succession.

2. Given $\left\{\begin{array}{r} 3x + \ \ 5y + \ \ 7z = \ \ 560 \\ 9x + 25y + 49z = 2920 \end{array}\right\}$, to find all the integral values of x, y, and z.

$$\text{Ans. } \left\{\begin{array}{l|l} z = 15 & 30 \\ y = 82 & 40 \\ x = 15 & 50. \end{array}\right.$$

PROBLEM VI.

To find the least whole number, which being divided by given numbers, shall leave given remainders.

Let a, a', a'', &c. be the given divisors, and b, b', b'', &c. the re-

spective remainders; also, call the required number N, then $N = ax + b = a'y + b' = a''z + b'' =$ &c.; therefore $ax - a'y = b' - b$; find then the least values of x and y in this equation, then will $ax + b$, or $a'y + b'$, be the least whole number that fulfils the two first conditions; call this number c, then it is obvious that this, and every other number fulfilling the same conditions, will be contained in the expression $aa'z' + c$,* z' being 0, 1, 2, &c. successively, we have, therefore, $aa'z' + c = a''z + b''$, to find the least values of z' and z, in which case $aa'z + c = a''z + b''$, will be the least whole number fulfilling the three first conditions; call this number d, then will this, and every other number fulfilling the same conditions, be contained in the expression $aa'a''y' + d$, and equating this with the fourth expression for the value of N, and deducing thence the least value of y', the expression $aa'a''y' + d$ will then be the least number answering the four first conditions; and so on to any proposed extent.

EXAMPLES.

1. Find the least whole number, which being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10 respectively.

Here $N = 11x + 3 = 19y + 5 = 29z + 10$,

and $\therefore\ 19y - 11x = -2$, and $y = \dfrac{11x - 2}{19}$;

$$\begin{array}{r}
11x - 2 \\
2 \\
\hline
22x - 4 \\
19x \phantom{{}-00} \\
\hline
3x - 4 \\
7 \\
\hline
21x - 28 \\
\hline
x + 24 \\
\hline
\end{array}$$

* We are here reasoning on the supposition that a, a', &c. are prime to each other; if, however, they have a common factor, it should be expunged from the expression $aa'z'$.

$\therefore x = 14$, and $11x + 3 = 157$; hence we have

$$11 \times 19x' + 157 = 209x' + 157 = 29z + 10,$$

$$\text{and } \therefore z = \frac{209x' + 147}{29} = 7x' + 5 + \frac{6x' + 2}{29};$$

$$\begin{array}{r} 6x' + 2 \\ 5 \\ \hline 30x' + 10 \\ 29x' \\ \hline x' + 10 \\ \hline \end{array}$$

$\therefore x' = 19$, and, consequently, $209x' + 157 = 4128$, the number required.

2. Find the least whole number, which being divided by 17 and 26, shall leave for remainders 7 and 13 respectively?

Ans. 143.

3. Find the least whole number, which being divided by 28, 19, and 15, shall leave for remainders 19, 15, and 11, respectively?

Ans. 7691.

4. Find the least whole number, which being divided by 3, 5, 7, and 2, shall leave for remainders 2, 4, 6, and 0, respectively?

Ans. 104.

5. Find the least whole number, which being divided by each of the nine digits, shall leave no remainders?

Ans. 2520.*

* For several particulars in this chapter the author is indebted to Barlow's Theory of Numbers, a work which cannot be too strongly recommended to the notice of the English student. There is no part of mathematical science that requires such an intimate acquaintance with the properties of numbers as the indeterminate analysis, and the work just mentioned is the only production on that interesting subject in the English language, with the exception of Malcolm's Arithmetic.

CHAPTER IX.

ON THE DIOPHANTINE ANALYSIS.

(168.) Diophantine Algebra* is that part of analysis which relates to the finding particular rational values for general expression under a surd form; the principal methods of effecting which are comprehended in the following problems.

PROBLEM I.

To find such values of x as will render rational the expression

$$\sqrt{ax^2+bx+c}.$$

Before we can give any direct investigation of this problem, it will be necessary to consider the nature of the known quantities a, b, c, because there are several cases in which the thing here proposed to be done becomes impossible, and that solely on account of these known quantities.

Case 1. *When $a=0$, or when the expression is of the form $\sqrt{bx+c}$.*

Put $\sqrt{bx+c}=p$, or $bx+c=p^2$, then $x=\dfrac{p^2-c}{b}$; consequently, whatever value be given to p, there must necessarily result a corresponding value of x that will render the proposed expression rational, and equal to p.

EXAMPLES.

1. Find a number such, that if it be multiplied by 5, and the product increased by 2, the result shall be a square.

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* So called from Diophantus, a Greek mathematician, who lived about 300 years after Christ, and who appears to have been the first writer on this branch of Algebra.

Put $5x + 2 = p^2$, then $x = \dfrac{p^2 - 2}{5}$; if we assume $p = 2$, then $x = \frac{2}{5}$; and by assuming other values for p, different values of x may be obtained.

2. Find two numbers, whose difference shall be equal to a given number a, and the difference of whose squares shall be also a square.

Let x be one number, then $a + x$ is the other, and we have to make $(a + x)^2 - x^2$, or $a^2 + 2ax$, a square.

Put $a^2 + 2ax = p^2$, then $x = \dfrac{p^2 - a^2}{2a}$, where the value of p may be any number assumed at pleasure.

3. Find a number such, that if it be multiplied by 9, and the product diminished by 7, the result shall be a square.

4. Find a number such, that if it be increased by $\frac{1}{3}$ of its own value, and 11 be taken from the sum, the remainder shall be a square.

CASE 2. *When $c = 0$, or when the expression is of the form* $\sqrt{ax^2 + bx}$.

Put $\sqrt{ax^2 + bx} = px$, or $ax^2 + bx = p^2x^2$, then $ax + b = p^2x$; whence $x = \dfrac{b}{p^2 - a}$, and whatever value be given to p in this expression, there will result a value of x that will make the proposed expression rational.

EXAMPLES.

Find a number such, that if its half be added to double its square, the result shall be a square.

Let x be the number, then we must have $2x^2 + \frac{1}{2}x =$ a square; which denote by p^2x^2, then $2x + \frac{1}{2} = p^2x$, or $2x - p^2x = -\frac{1}{2}$; $\therefore x = \dfrac{\frac{1}{2}}{p^2 - 2}$, p being any number whatever. If p be taken $= 2$, then $x = \frac{1}{4}$.

2. Find two numbers, whose sum shall be equal to a given number a, and whose product shall be a square.

Let x be one number, then $a - x$ is the other, and we have to make $ax - x^2$ a square. Put $ax - x^2 = p^2x^2$, then $a - x = p^2x$, whence $x = \dfrac{a}{p^2+1}$, p being any number whatever.

3. Find a number such, that if its square be multiplied by 7, and the number itself by 8, the sum of the products shall be a square.

4. Find a number such, that if its square be divided by 10, and the number itself by 3, the difference of the quotients shall be a square.

CASE 3. *When c is a square, or when the expression is of the form*
$$\sqrt{ax^2 + bx + c^2}.$$

Put $\sqrt{ax^2 + bx + c^2} = px + c$, then $ax^2 + bx + c^2 = p^2x^2 + 2cpx + c^2$, or $ax^2 + bx = p^2x^2 + 2cpx$, $\therefore\ ax + b = p^2x + 2cp$, whence $x = \dfrac{2cp - b}{a - p^2}$.

EXAMPLES.

Find two numbers, whose sum shall be 16, and such, that the sum of their squares shall be a square.

Let x be one number, then $16 - x$ is the other, and we have to make $x^2 + (16 - x)^2$, or $2x^2 - 32x + 256$, a square, which denote by $(px - 16)^2 = p^2x^2 - 32px + 256$, and we then have $2x^2 - 32x = p^2x^2 - 32px$, or
$$2x - 32 = p^2x - 32p, \text{ whence } x = \frac{32(p-1)}{p^2-2}.$$

If we take $p = 3$, we shall have $x = 9\frac{1}{7}$, $\therefore$ the two numbers are $9\frac{1}{7}$, and $6\frac{6}{7}$.

2. Find two numbers, whose difference shall be equal to a given number a, and the sum of whose squares shall be a square.

CASE 4. *When a is a square, or when the expression is of the form*
$$\sqrt{a^2x^2 + bx + c}.$$

Put $\sqrt{a^2x^2 + bx + c} = ax + p$, or
$$a^2x^2 + bx + c = a^2x^2 + 2pax + p^2,$$
then $bx + c = 2pax + p^2$ $\therefore x = \dfrac{c - p^2}{2pa - b}$.

EXAMPLES.

1. Find a number such, that if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let x be the number, then we have to make $(x+2)(x+5)$, or $x^2+7x+10$, a square, which denote by $(x-p)^2$, then $x^2+7x+10 = x^2-2px+p^2$, or $7x+10 = -2px+p^2$, $\therefore x = \dfrac{p^2-10}{7+2p}$.

If we take $p = 4$, we shall have $x = \frac{2}{5}$.

2. Find two numbers, whose difference shall be 14, and such, that if the first be increased by 3, and the second by 4, the product of the sums shall be a square.

3. Find two numbers, whose difference shall be 3, such, that if twice the first increased by 3, be multiplied by twice the second diminished by 3, the product shall be a square.

Case 5. *When neither a nor c are squares, but when b^2-4ac is a square.*

In this case it will first be necessary to show that the expression ax^2+bx+c will always be resolvable into two possible factors. For if we put $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, and solve the equation, or find the two values of x in it, as $x=k$, and $x=k'$, then $x-k$, and $x-k'$, will obviously be the two factors of $x^2+\dfrac{b}{a}x+\dfrac{c}{a}$; and therefore $a(x-k)(x-k')$ will be equal to the proposed expression.

Now the values of x in the above equation are

$$x = -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}, \text{ and } x = -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$$

or putting $b^2-4ac=d^2$, the values of x are

$$\frac{d-b}{2a}, \text{ and } -\frac{b+d}{2a}, \text{ and, consequently,}$$

$$\left(ax + \frac{b-d}{2}\right)\left(x + \frac{b+d}{2a}\right) = ax^2 + bx + c;$$

we see therefore that the proposed expression under these conditions is always resolvable into two factors.

Let there be then

$$\sqrt{ax^2 + bx + c} = \sqrt{(fx+g)(hx+k)},$$

which put equal to $p(fx+g)$, then

$$(fx+g)(hx+k) = p^2(fx+g)^2;$$

$$\text{or } (hx+k) = p^2(fx+g);$$

$$\text{whence } x = \frac{p^2g - k}{h - p^2f}.$$

EXAMPLES.

1. Find such a value of x as will render the expression $6x^2 + 13x + 6$ a square.

Here $a = 6$, $b = 13$, and $c = 6$, and, as this expression evidently does not belong to any of the preceding cases, it will be proper to try whether $b^2 - 4ac$ is a square, which it is found to be, viz. 25: we are certain, therefore, that the expression may be represented by two factors, which are readily found to be $2x + 3$, and $3x + 2$.

Put therefore $6x^2 + 13x + 6$, or $(2x+3)(3x+2) = [p(2x+3)]^2$, and it follows that $3x + 2 = p^2(2x+3)$,

$$\text{whence } x = \frac{3p^2 - 2}{3 - 2p^2}.$$

If we take $p = 1$, then $x = 1$, and the expression becomes equal to 25.

2. Find such a value of x as will make $2x^2 + 10x + 12$ a square.

3. Find such a value of x as will render rational the expression $\sqrt{8x^2 + 6x - 2}$.

Case 6. *When the proposed expression can be divided into two parts, one of which is a square, and the other the product of two factors.*

This is the last case in which any general method of proceeding can be pointed out, and may often be serviceable when the expression

does not come under either of the preceding cases. It is, however, sometimes troublesome, to find whether the proposed expression can be decomposed as this case requires, or not; but if it be ascertained that it can, the expression $\sqrt{ax^2 + bx + c}$ may be put under the form $\sqrt{(dx + e)^2 + (fx + g)(hx + k)}$, and if we equate this with $(dx + e) + p(fx + g)$, there will result

$$(dx + e)^2 + (fx + g)(hx + k)$$
$$= (dx + e)^2 + 2p(dx + e)(fx + g) + p^2(fx + g)^2,$$
$$\text{or } hx + k = 2p(dx + e) + p^2(fx + g);$$
$$\text{whence } x = \frac{p(2e + pg) - k}{h - p(2d + pf)}.$$

EXAMPLES.

Find a value of x such, that $2x^2 + 8x + 7$ shall be a square.

This expression, after a few trials, is found to be equivalent to $(x+2)^2 + (x + 1)(x + 3)$, which being equated with
$[(x + 2) - p(x + 1)]^2 = (x + 2)^2 - 2p(x + 2)(x + 1) + p^2(x + 1)^2$, there results $x + 3 = -2p(x + 2) + p^2(x + 1)$;

$$\text{whence } x = \frac{p^2 - 4p - 3}{1 + 2p - p^2}.$$

If we take $p = 3$, we shall have $x = 3$, and

$$2x^2 + 8x + 7 = 49.$$

2. Find a value of x such, that $12x^2+17x+6$ may be a square.

(190.) We have now given all the cases in which general methods have been discovered to render the expression $\sqrt{ax^2+bx+c}$ rational; but as it may have rational values in other cases, it is of importance to be able to determine them.

Now this can only be done when one satisfactory value is already known, which value must therefore be found by trial; this being obtained, other values may be readily deduced.

(169.) Suppose the expression $\sqrt{ax^2+bx+c}$ is found to become rational when $x = r$, and that the value of the expression in this

case is s; then $ar^2 + br + c = s^2$. Put $x = y + r$, and we have, by substitution,

$$ax^2 + bx + c = a(y+r)^2 + b(y+r) + c$$
$$= ay^2 + (2ar + b)y + ar^2 + br + c;$$
$$= ay^2 + (2ar + b)y + s^2,$$

and, as this form comes under Case 3, the value of y, in order that this last expression may be a square, can be found, and thence that of $x = y + r$.

EXAMPLES.

1. Find such values of x that will render the expression $\sqrt{10 + 8x - 2x^2}$ rational.

This expression is found to become rational when $x = 3$.

Put therefore $x = 3 + y$, and we have, by substitution, $10 + 8x - 2x^2 = 16 - 4y - 2y^2$, which must be a square; denote it by $(4 - py)^2 = 16 - 8py + p^2y^2$, and we shall have

$$16 - 4y - 2y^2 = 16 - 8py + p^2y^2,$$
$$\text{or } -4 - 2y = -8p + p^2y;$$
$$\text{whence } y = \frac{8p - 4}{p^2 + 2}:$$

If we take $p = 2$, then $y = 2$, and $\therefore x = 5$, and the value of the proposed expression is 0.

2. Find such values of x as will render the expression $\sqrt{5x^2 + 12x + 8}$ rational.

3. Find a number such, that if three times itself be taken from three times its square, the remainder increased by 3 shall be a square.

PROBLEM II.

To find such values of x as will render rational the expression

$$\sqrt{ax^3 + bx^2 + cx + d}.$$

There are but two cases in which a direct solution can be given to this problem. These are the following:

CASE 1. *When the two last terms are absent, or when the expression is of the form*

$$\sqrt{ax^3 + bx^2}.$$

Put $\sqrt{ax^3 + bx^2} = px$, or $ax^3 + bx^2 = p^2x^2$, then $ax + b = p^2$;

$$\text{whence } x = \frac{p^2 - b}{a}.$$

EXAMPLES.

1. Find a number such, that if three times its cube be added to twice its square, the sum shall be a square.

Here we must make $3x^3 + 2x^2$ a square; let p^2x^2 be the square, then $3x + 2 = p^2$, $\therefore\ x = \dfrac{p^2 - 3}{2}$.

If we take $p = 3$, we have $x = 3$, the number required.

2. Find a number such, that if five times its square be taken from three times its cube, the remainder shall be a square.

CASE 2. *When the last term is a square, or when the expression is of the form*

$$\sqrt{ax^3 + bx^2 + cx + d^2}.$$

Put $\sqrt{ax^3 + bx^2 + cx + d^2} = \dfrac{c}{2d}x + d$*;

then $ax^3 + bx^2 + cx + d^2 = \dfrac{c^2}{4d^2}x^2 + cx + d^2$,

$$\text{or } ax^3 + bx^2 = \frac{c^2}{4d^2}x^2;$$

$$\therefore\ ax + b = \frac{c^2}{4d^2};$$

$$\text{whence } x = \frac{c^2 - 4bd^2}{4ad^2}.$$

---

* The expression is assumed equal to $\dfrac{c}{2d}x + d$, in order that the two last terms in its square may be the same as the corresponding terms in the proposed expression.

This solution gives only one value of x, but from this, other values, when possible, may be obtained by the method next following

When the second and third terms are absent, this method evidently fails.

EXAMPLES.

1. Find such a value of x as will make the expression $3x^3 - 5x^2 + 6x + 4$ a square.

Put $3x^3 - 5x^2 + 6x + 4 = (\frac{3}{2}x + 2)^2 = \frac{9}{4}x^2 + 6x + 4$,

$$\text{then } 3x^3 - 5x^2 = \tfrac{9}{4}x^2, \text{ or } 3x - 5 = \tfrac{9}{4};$$

$$\text{whence } x = \tfrac{29}{12},$$

which value being substituted in the proposed expression, makes it equal to $(\frac{45}{8})^2$.

2. Find such a value of x as will make $x^3 - x^2 + 2x + 1$ a square.

Ans. $x = 2$.

3. Find a value of x that will make the expression $-5x^3 + 6x^2 - 4x + 1$ a square.

Ans. $x = -1$.

To these two cases may be added, as in the last Problem, a third, by which other values may be had from one being previously known.

(170.) Suppose it is already known that the expression

$$\sqrt{ax^3 + bx^2 + cx + d}$$

becomes rational when $x = r$, and that the value of the expression then becomes $= s$; that is, let

$$ar^3 + br^2 + cr + d = s^2;$$

then, as in Art. (191), put $x = y + r$, and we have

$$\begin{aligned} ay^3 + 3ary^2 + 3ar^2y + ar^3 &= ax^3 \\ by^2 + 2bry + br^2 &= bx^2 \\ cy + cr &= cx \\ d &= d \\ \hline ay^3 + b'y^2 + c'y + s^2 &= \square^*, \end{aligned}$$

* This symbol is used to signify the words, *a square.*

b', c', and s^2, representing the sums of the quantities under which they are respectively placed, therefore the value of y may be determined by last case.

EXAMPLES.

The expression $\sqrt{x^3 - x^2 + 2x + 1}$ is found to become rational when $x = 2$: it is required to find another value of x that will answer.

Put $x = y + 2$, then $x^3 - x^2 + 2x + 1 = y^3 + 3y^2 + 8y + 9$; assume this last expression equal to

$$(\tfrac{4}{3}y + 3)^2, \text{ or } \tfrac{16}{9}y^2 + 8y + 9;$$
$$\text{then } y^3 + 3y^2 = \tfrac{16}{9}y^2, \text{ or } y + 3 = \tfrac{16}{9};$$
$$\text{whence } y = -\tfrac{11}{9}, \text{ and } \therefore x = 2 + y = \tfrac{7}{9}.$$

2. Find a value of x in the expression $\sqrt{x^3 + 3} = \square$, besides the case $x = 1$.

Ans. $x = -\frac{23}{16}$.

3. Find a value of x in the expression $\sqrt{3x^3 + 1} = \square$, besides the case $x = 1$.

Ans. $x = -\frac{5}{16}$.

SCHOLIUM.

There are many cases in the preceding Problem in which the unknown quantity admits of only one rational value, and many more in which the expression is impossible. If any expression can be divided into factors, one of which is a square, this square may be rejected, and the remaining factors only used. Thus, if the expression $ax^3 + bx^2$, or $x^2(ax + b)$, is to be made a square, it will only be necessary to make $ax + b$ a square; also, in the expression $x^3 - x^2 - x + 1$, which is equal to $(1 - x)^2(1 + x)$, it will be only necessary to make $1 + x$ a square, in order that the whole expression may be a square.

PROBLEM III.

To find such values of x as will render rational the expression

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}.$$

In this Problem there are three cases in which a direct solution can be obtained.

CASE 1. *When both the first and last terms are complete squares, or when the expression is of the form*

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e^2}.$$

Put $a^2x^4+bx^3+cx^2+dx+e^2=(ax^2+mx+e)^2=a^2x^4+2amx^3+(m^2+2ae)x^2+2mex+e^2$;

then, in order that the three first terms in each side of this equation may destroy each other, we must make

$$b = 2am, \text{ or } m = \frac{b}{2a},$$

and there will result

$$cx^2 + dx = (m^2 + 2ae)x^2 + 2mex;$$

$$\text{whence } x = \frac{d - 2me}{m^2 + 2ae - c},$$

or, substituting for m its equal $\frac{b}{2a}$, we have

$$x = \frac{4a(ad - be)}{b^2 + 4a^2(2ae - c)};$$

or, since e is found in the proposed expression only in its second power, it may be taken either positively or negatively; hence we get another value of x, viz.

$$x = \frac{4a(ad + be)}{b^2 - 4a^2(2ae + c)}.$$

Or this case of the problem may be solved differently by making $d = 2me$, when m will be equal to $\frac{d}{2e}$, instead of $\frac{b}{2a}$, and we shall have

$$bx^3 + cx^2 = 2amx^3 + (m^2 + 2ae)x^2;$$

$$\text{whence } x = \frac{m^2 + 2ae - c}{b - 2am};$$

or, substituting for m its equal $\frac{d}{2e}$, we have

$$x = \frac{d^2 + 4e^2(2ae - c)}{4e(be - ad)};$$

$$\text{or } x = \frac{d^2 - 4e^2(2ae + c)}{4e(be + ad)};$$

this last value being obtained from supposing e negative, as before.

Hence, by employing these two methods, four solutions may be obtained: it must be observed, however, that they all fail when b and d are both 0.

EXAMPLES.

1. It is required to find such a value of x, that the expression $x^4 - 6x^3 + 4x^2 - 24x + 16$ may be a square.

Put, according to the first of the above methods,

$$x^4 - 6x^3 + 4x^2 - 24x + 16 = (x^2 - 3x - 4)^2$$
$$= x^4 - 6x^3 + x^2 + 24x + 16,$$

and there results

$$4x^2 - 24x = x^2 + 24x,$$
$$\text{or } 4x - 24 = x + 24;$$
$$\text{whence } x = \tfrac{48}{3} = 16.$$

If, according to the second method, we put the expression equal to

$$(x^2 + 3x - 4)^2 = x^4 + 6x^3 + x^2 - 24x + 16,$$
$$\text{we have } 6x^3 + x^2 = -6x^3 + 4x^2;$$
$$\text{whence } x = \tfrac{1}{4}.$$

By taking $4\ (= e)$ positive, each of these solutions gives $x = 0$.

2. It is required to find such values of x as will make $x^4 - 2x^3 + 2x^2 + 2x + 1$ a square.

Ans. $x = 4$, or $-\frac{1}{4}$.

3. It is required to find such values of x as will make $4x^4 + 3x + 1$ a square.

Ans. $x = \frac{228}{576}$, or $\frac{447}{576}$.

CASE 2. *When the first term only is a square, or when the expression is of the form*

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e}.$$

Put $a^2x^4 + bx^3 + cx^2 + dx + e = (ax^2 + mx + n)^2 =$
$ax^4 + 2amx^3 + (m^2 + 2an)x^2 + 2mnx + n^2$;

then, in order that the first three terms in this equation may destroy each other, we must make

$$\left.\begin{array}{l} b = 2am \\ c = m^2 + 2an \end{array}\right\} \text{whence} \left\{\begin{array}{l} m = \dfrac{b}{2a} \\ n = \dfrac{c - m^2}{2a} = \dfrac{4a^2c - b^2}{8a^3}, \end{array}\right.$$

we have therefore $dx + e = 2mnx + n^2$;

$$\text{whence } x = \frac{n^2 - e}{d - 2mn};$$

or, substituting for m and n their values as deduced above, we have

$$x = \frac{(4a^2c - b^2)^2 - 64a^6e}{8a^2[8a^4d - b(4a^2c - b^2)]};$$

When b and d are both 0, this formula fails, the same as in the last case.

EXAMPLES.

1. Required a value of x such, that the expression
$4x^4 + 4x^3 + 4x^2 + 2x - 6$ may become a square.

Here $m = 1$, and $n = \frac{3}{4}$, therefore

put $4x^4 + 4x^3 + 4x^2 + 2x - 6 = (2x^2 + x + \frac{3}{4})^2 =$
$4x^4 + 4x^3 + 4x^2 + \frac{3}{2}x + \frac{9}{16}$,

and we have $2x - 6 = \frac{3}{2}x + \frac{9}{16}$;

whence $x = \frac{105}{8} = 13\frac{1}{8}$.

2. Required such a value of x, that the expression $x^4 - 3x + 2$ may become a square.

Ans. $x = \frac{2}{3}$.

3. Required such a value of x, that the expression $x^4 - 2x^3 + 4x^2 - 2x + 2$ may be a square.

Ans. $x = \frac{1}{4}$.

CASE 3. *When the last term only is a square, or when the expression is of the form*

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e^2}$$

Put $ax^4 + bx^3 + cx^2 + dx + e^2 = (mx^2 + nx + e)^2 =$

$$m^2x^4 + 2mnx^3 + (n^2 + 2me)x^2 + 2nex + e^2;$$

then, in order that the three last terms on each side of this equation may destroy each other, we must make

$$\left.\begin{aligned} d &= 2ne \\ c &= n^2 + 2me \end{aligned}\right\} \text{whence} \left\{\begin{aligned} n &= \frac{d}{2e} \\ m &= \frac{e - n^2}{2e} = \frac{4ce^2 - d^2}{8e^3}, \end{aligned}\right.$$

and we shall then have

$$ax^4 + bx^3 = m^2x^4 + 2mnx^3,$$

$$\text{or } ax + b = m^2x + 2mn;$$

$$\text{whence } x = \frac{2mn - b}{a - m^2};$$

or, substituting for m and n, their values as deduced above, we have

$$x = \frac{8e^2[d(4ce^2 - d^2) - 8be^4]}{64ae^6 - (4ce - d^2)^2},$$

which formula fails under the same circumstances as those of the preceding cases.

The first case of this Problem is evidently included in each of the two last cases, and therefore either of the two formulæ last deduced is also applicable to the first case.

EXAMPLES.

1. Find such a value of x as will make the expression

$$5x^4 - 4x^3 + 3x^2 - 2x + 1 \text{ a square.}$$

Here $m = 1$, and $n = -1$, therefore

$$\text{put } 5x^4 - 4x^3 + 3x^2 - 2x + 1 = (x^2 - x + 1)^2 =$$

$$x^4 - 2x^3 + 3x^2 - 2x + 1,$$

$$\text{and we have } 5x^4 - 4x^3 = x^4 - 2x^3,$$
$$\text{or } 5x - 4 = x - 2;$$
$$\text{whence } x = \tfrac{2}{4} = \tfrac{1}{2}.$$

2. Find a value of x such, that we may have $2x^4 - 3x + 1 = \square$.

Ans. $x = \frac{216}{47}$.

3. Find such a value of x that we may have $22x^4 - 40x^3 - 40x^2 + 64x + 16 = \square$.

Ans. $x = \frac{6}{7}$.

When the proposed expression does not come under either of the above cases, then, as in the preceding Problems, one satisfactory value of the unknown quantity must be discovered by trial, after which, other values, when possible, may be obtained; but in this, as well as in the preceding Problems, there are many expressions in which the unknown quantity admits of only one value, and, in a great many instances, the value is impossible.* We now proceed to show how to find other values from having one value already given.

(171.) Suppose it is already known that the expression

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}$$

becomes rational when $x = r$, and that we have

$$ar^4 + br^3 + cr^2 + dr + e = s^2.$$

Assume $y + r = x$, and we have

$$\begin{array}{rcl}
ay^4 + 4ary^3 + 6ar^2y^2 + 4ar^3y + ar^4 & = & ax^4 \\
by^3 + 3bry^2 + 3br^2y + br^3 & = & bx^3 \\
cy^2 + 2cry + cr^2 & = & cx^2 \\
dy + dr & = & dx \\
e & = & e \\
\hline
ay^4 + b'y^3 + c'y^2 + d'y + s^2 & = & \square;
\end{array}$$

the terms in the last line representing the sums of the quantities under which they are respectively placed.

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* It would be impracticable to give in this work a view of all the impossible forms of the expression here treated of: the reader is therefore referred to the work mentioned at the conclusion of last chapter.

Hence the expression is reduced to a form in which the preceding case will apply, and therefore the value of y, and thence that of x, may be determined.

EXAMPLES.

1. Find such values of x, that the expression

$$3x^4+2x^3-5x^2+7x-3 \text{ may be a square.}$$

It appears, upon trial, that if 1 be substituted for x, the expression will become a square, viz. 4.

Put therefore $x=y+1$, and we have

$$3x^4+2x^3-5x^2+7x-3=3y^4+14y^3+19y^2+15y+4,$$

which must be made a square; therefore, according to the last case, denote this square by

$$(\tfrac{79}{64}y^2+\tfrac{15}{4}y+2)^2=\tfrac{6241}{4096}y^4+\tfrac{1185}{128}y^3+19y^2+15y+4,$$

and we shall then have

$$3y^4+14y^3=\tfrac{6241}{4096}y^4+\tfrac{1185}{128}y^3,$$

$$\text{or } 3y+14=\tfrac{6241}{4096}y+\tfrac{1185}{128};$$

$$\text{whence } y=\tfrac{19424}{6047},$$

$$\text{and, consequently, } x=\tfrac{25471}{6047}.$$

2. Find a value of x that will make $\sqrt{x^4-2x^3+2}$ rational, besides the case $x=1$.

Ans. $x=\frac{1}{3}$.

3. Find a value of x such, that the expression

$$22x^4-128x^3+212x^2-64x-26$$

may be a square, the case $x=1$ being already known.

Ans. $x=\frac{15}{7}$.*

PROBLEM IV.

To find such values of x as will render rational the expression

$$\sqrt[3]{ax^3+bx^2+cx+d}.$$

---

* No methods have yet been discovered for rendering expressions of the above kind rational squares, if the unknown quantity exceed the fourth power; not even when a satisfactory case has been obtained by trial.

In this Problem there are likewise only three cases in which a direct solution can be obtained. These are as follow.

CASE 1. ***When both first and last terms are cubes, or when the expression is of the form***

$$\sqrt[3]{a^3x^3 + bx^2 + cx + d^3}.$$

Put $a^3x^3 + bx^2 + cx + d^3 = (ax + d)^3 = a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3$, and we have

$$bx^2 + cx = 3a^2dx^2 + 3ad^2x,$$

$$\text{or } bx + c = 3a^2dx + 3ad^2;$$

$$\text{whence } x = \frac{3ad^2 - c}{b - 3a^2d}.$$

EXAMPLES.

1. Find a value of x such, that the expression

$$x^3 + 9x^2 + 4x + 8 \text{ may be a cube.}$$

Put $x^3 + 9x^2 + 4x + 8 = (x + 2)^3 = x^3 + 6x^2 + 12x + 8$, and we shall then have

$$9x^2 + 4x = 6x^2 + 12x$$

$$\text{whence } x = \tfrac{8}{3} = 2\tfrac{2}{3}.$$

2. Find a value of x such, that the expression $-125x^3 + 89x^2 + 28x + 8$ may be a cube.

Ans. $x = \frac{88}{61}$.

3. Find a value of x such, that the expression $8x^3 + 42x^2 - 8x + 27$ may be a cube.

Ans. $x = 10\frac{1}{3}$.

CASE 2. ***When the first term only is a cube, or when the expression is of the form***

$$\sqrt[3]{a^3x^3 + bx^2 + cx + d}.$$

Put $a^3x^3 + bx^2 + cx + d = (ax + m)^3 = a^3x^3 + 3a^2mx^2 + 3am^2x + m^3$, and make

$$3a^2m = b, \text{ or } m = \frac{b}{3a^2},$$

Assume $y + r = x$, and we have

$$\begin{aligned} ay^3 + 3ary^2 + 3ar^2y + ar^3 &= ax^3 \\ by^2 + 2bry + br^2 &= bx^2 \\ cy + cr &= cx \\ d &= d \\ \hline ay^3 + b'y^2 + c'y + s^3 &= \text{a cube.} \end{aligned}$$

The expression is therefore reduced to a form which is resolvable by last case.

EXAMPLES.

1. It is required to find such values for x, that the expression $2x^3 - 4x^2 + 6x + 4$ may be a cube.

It appears, upon trial, that $x = 1$ is a satisfactory value; put then $x = y + 1$, and the expression becomes

$$2y^3 + 2y^2 + 4y + 8,$$

which put equal to

$$(\tfrac{1}{3}y + 2)^3 = \tfrac{1}{27}y^3 + \tfrac{2}{3}y^2 + 4y + 8,$$

and there results

$$2y^3 + 2y^2 = \tfrac{1}{27}y^3 + \tfrac{2}{3}y^2,$$

$$\text{or } 2y + 2 = \tfrac{1}{27}y + \tfrac{2}{3};$$

$$\text{whence } y = -\tfrac{36}{53};$$

$$\text{and, consequently, } x = \tfrac{17}{53}.$$

2. Find a value of x that will make $x^2 + x + 1$ a cube, besides the case $x = -1$.

Ans. $x = -19$.

3. Find such a value of x, that the expression $2x^3 - 1$ may be a cube, besides the case $x = 1$.

Ans. Impossible.

ON DOUBLE AND TRIPLE EQUALITIES.

(173.) In the preceding Problems, the object of our investigations has been to find rational values for expressions under a surd form; and our inquiries have been directed to each expression separately. Questions, however, often occur in the diophantine analysis, that require us to find values for the unknown quantity, or quantities, that shall not only render a single expression a square, cube, &c., but that shall also, at the same time, fulfil similar conditions in one or more other expressions, containing the same unknown quantity or quantities. In the case where two expressions are concerned, it is called a *double equality*, and where there are three expressions, a *triple equality*, &c. The following methods of resolving these equalities will be of service to the student in ordinary cases; but in those instances where the methods here given are found to be insufficient, he must be guided by his own penetration and ingenuity, since no general method of proceeding, that shall be suitable to every case that may occur, can be given.

PROBLEM I.

To resolve the double equality

$$ax + b = \square,$$
$$cx + d = \square.$$

Put $ax+b=p^2$, and $cx+d=q^2$, then, equating the two values of x, which these equations furnish, we have

$$\frac{p^2-b}{a} = \frac{q^2-d}{c}, \text{ or } cp^2 - cb = aq^2 - ad;$$

$$\text{therefore } c^2p^2 = caq^2 - cad + c^2b,$$

and, consequently, q must be such a value that the expression

$$caq^2 - cad + c^2b$$

may become a square, which value may be ascertained by one or other of the preceding methods, and thence the value of x may be determined.

PROBLEM II.

To resolve the double equality

$$ax^2 + bx = \square,$$
$$cx^2 + dx = \square.$$

Put $x = \frac{1}{y}$, then, if each equality be multiplied by y^2, there will result the double equality

$$a + by = \square,$$
$$c + dy = \square,$$

which belongs to the preceding Problem.

Or put $ax^2 + bx = p^2x^2$, then $ax + b = p^2x$, and, consequently, $x = \frac{b}{p^2 - a}$, and $\therefore cx^2 + dx = c(\frac{b}{p^2-a})^2 + d(\frac{b}{p^2-a}) = \square$; or, multiplying by the square $(p^2 - a)^2$, it becomes

$$cb^2 - abd + bdp^2 = \square;$$

whence p may be determined, and thence x.

PROBLEM III.

To resolve the double equality

$$ax^2 + bx + c = \square,$$
$$dx^2 + ex + f = \square.$$

Here it will be necessary first to resolve the equality

$$ax^2 + bx + c = \square$$

by Problem 1, and to substitute the value of x so deduced in the second equality

$$dx^2 + ex + f = \square,$$

which will, in consequence, rise to the fourth power, and therefore its solution will belong to Prob. III., p. 257.

PROBLEM IV.

To resolve the triple equality

$$ax + by = \square,$$
$$cx + dy = \square,$$
$$ex + fy = \square.$$

Put

$$ax + by = t^2,$$
$$cx + dy = u^2,$$
$$ex + fy = s^2;$$

then, by expunging y from the two first equations, we have

$$x = \frac{dt^2 - bu^2}{ad - bc};$$

and, by expunging x from the same equations, we have

$$y = \frac{au^2 - ct^2}{ad - bc};$$

therefore, by substituting for x and y, in the third equation, their respective values here exhibited, we shall have

$$\frac{af - be}{ad - bc}u^2 - \frac{cf - de}{ad - bc}t^2 = \square;$$

or putting $u = tz$, and dividing the expression by the square t^2, there arises the equality

$$\frac{af - be}{ad - bc}z^2 - \frac{cf - de}{ad - bc} = \square;$$

from which the values of z may be determined.

Having then found the values of z, we shall have, from the above values of x and y, observing to write tz for u, the following results, viz.

$$x = \frac{d - bz^2}{ad - bc}t^2, \text{ and } y = \frac{az^2 - c}{ad - bc}t^2,$$

where t may be any value whatever.

(174.) The above are the most general methods hitherto discovered for the resolution of double and triple equalities; we may therefore proceed to show the practical application of the foregoing

parts of the present chapter to the solution of diophantine questions: but, as has been already said, the student must be expected to meet with cases in which the mode of proceeding must be left, in a great measure, for his own penetration and judgment to suggest. Indeed, the subject on which we are now treating has exercised the ingenuity of some of the most eminent mathematicians of Europe; but Euler and Lagrange have been the most successful in combating the difficulties with which it is attended. The performances of the former are contained in the second volume of his Algebra, which, with the additions of Lagrange, forms the most complete body of information on the diophantine analysis extant; and it is to this work chiefly that the attention of the student is directed.* In the following solutions it will frequently be observed that much depends upon the nature and relation of the assumptions made at the commencement, as a little artifice and ingenuity here will often enable us readily to satisfy one or two conditions of the question, when those that remain may be fulfilled by one or other of the known methods already given.

MISCELLANEOUS DIOPHANTINE QUESTIONS.

QUESTION I.

It is required to find a number such, that if it be either increased or diminished by a given number a, and the result be multiplied by the number sought, the product shall, in either case, be a square.

Let x be the number required, then we have to make

$$x^2 + ax = \square,$$
$$x^2 - ax = \square.$$

Put $x = \frac{1}{y}$, then these expressions become

* The reader is also referred to Barlow's Theory of Numbers; to Leybourn's Mathematical Repository; to the masterly papers of Mr. Cunliffe, in different volumes of the Gentleman's Mathematical Companion; and to a paper, by the late Professor Leslie, in Vol. II. of the Edinburgh Philosophical Transactions.

$$\frac{1}{y^2}+\frac{a}{y}=\square,$$

$$\frac{1}{y^2}-\frac{a}{y}=\square,$$

and, multiplying each by y^2, we shall have to make $1+ay$, and $1-ay$, squares; in order to which, put $1+ay=p^2$, and we get $y=\dfrac{p^2-1}{a}$, and therefore by substitution,

$$1-ay=1-p^2+1=2-p^2=\square.$$

Now in this last expression we must first find a satisfactory value of p by trial, which is readily effected, since $p=1$ succeeds: assume, therefore, $p=1-q$, and then

$$2-p^2=1+2q-q^2=\square,$$

which denote by

$$(1-rq)^2=1-2rq+r^2q^2,$$

and we get

$$2-q=r^2q-2r,$$

$$\text{and, consequently, } q=\frac{2r+2}{r^2+1};$$

$$\text{whence } x=\frac{1}{y}=\frac{a}{q^2-2q}=\frac{a(1+r^2)^2}{4r(1-r^2)};$$

where r may be any number whatever; and, should any of the resulting values of x be negative, they may, with equal truth, be taken positively, as the proposed conditions will evidently obtain in either case.

Suppose $r=2$, and $a=1$, then $x=-\frac{25}{24}$, or $+\frac{25}{24}$. If $a=2$, then $x=\frac{25}{12}$; and so on for other values.

The former part of the above solution might have been conducted differently; thus,

Put $x^2+ax=p^2x^2$, then $x+a=p^2x$, or

$$x=\frac{a}{p^2-1}; \text{ whence, by substitution,}$$

$$x^2-ax=\left(\frac{a}{p^2-1}\right)^2-a\left(\frac{a}{p^2-1}\right)=\square,$$

or, multiplying by $(p^2-1)^2$, we have

$$a^2-a^2p^2+a^2=2a^2-a^2p^2=\square,$$

and dividing by a^2, there results $2-p^2=\square$, as before.

QUESTION II.

It is required to find three numbers in arithmetical progression such, that the sum of every two of them may be a square.

Let x, $x+y$, and $x+2y$ represent the three numbers, and put

$$2x+\ y=t^2,$$
$$2x+2y=u^2,$$
$$2x+3y=s^2;$$

then, exterminating x from the two first of these equations, we obtain

$$\frac{t^2-y}{2}=\frac{u^2-2y}{2},$$

from which we get $y=u^2-t^2=s^2-u^2$, and thence $2u^2-t^2=s^2$.

Put now $u=tz$, and this last equation becomes

$$2t^2z^2-t^2=s^2,$$

therefore $2z^2-1=\frac{s^2}{t^2}$; hence, $2z^2-1$ must be a square, which we find to be the case when $z=1$, therefore, putting $z=1-p$, we have

$$2z^2-1=1-4p+2p^2=\square,$$

which denote by

$$(1-rp)^2=1-2rp+r^2p^2,$$

and we have

$$-4p+2p^2=-2rp+r^2p^2,$$

from which we get $p=\frac{2r-4}{r^2-2}$,

and thence $z=1-p=\frac{r^2-2r+2}{r^2-2}$;

where r may be any number whatever; and, after having determined z, we shall obtain the values of x and y from the equations

$$x=\tfrac{1}{2}(t^2-y)=\tfrac{1}{2}(2-z^2)t^2,$$

$$\text{and } y=u^2-t^2=(z^2-1)t^2,$$

t also being any assumed number. In order that x and y may be positive, it is evident that z must lie between 1 and $\sqrt{2}$.

By taking $r = \frac{2}{3}$, we shall have

$$z = \tfrac{41}{31}, \therefore\ x = \frac{241}{2\,(31)^2}t^2, \text{ and } y = \frac{720}{(31)^2}t^2;$$

and making $t = 2 \times 31$, we have $x = 482$, and $y = 2880$; therefore 482, 3362, and 6242, are the numbers required.

QUESTION III.

Find two numbers such, that if to each, as also to their sum, a given square, a^2, be added, the three sums shall all be squares.

Let the two numbers be represented by $x^2 - a^2$, and $y^2 - a^2$, and then the two first conditions will be satisfied, and therefore it remains only to make

$$x^2 + y^2 - 2a^2 + a^2, \text{ or } x^2 + y^2 - a^2, \text{ a square,}$$

which denote by m^2, then

$$x^2 - a^2 = m^2 - y^2, \text{ or } (x + a)(x - a) = (m + y)(m - y).$$

Put $x + a = p\,(m - y)$, then $x - a = \dfrac{m + y}{p}$,

$$\text{whence } x = p\,(m - y) - a = \frac{m + y}{p} + a,$$

$$\text{and, consequently, } y = \frac{p^2m - 2ap - m}{p^2 + 1}.$$

Suppose $a = 1$, $p = 2$, and $m = 8$, then $y = 4$, and $x = 7$.

QUESTION IV.

Find three squares, whose sum shall be a square.

Let the three squares be x^2, y^2, and z^2; then

$$x^2 + y^2 + z^2 = \square.$$

Put $y^2 = 2xz$, or $x = \dfrac{y^2}{2z}$, and the expression becomes

$$x^2 + 2xz + z^2,$$

which is obviously a square, y and z being any assumed numbers.

If we take $y = 4$, and $z = 8$, then $x = 1$, and

$$1 + 16 + 64 = 81.$$

Otherwise, assume

$$x^2 + y^2 + z^2 = (x + p)^2 = x^2 + 2px + p^2,$$

and we shall then have

$$x = \frac{y^2 + z^2 - p^2}{2p}:$$

If we take $y = 4$, $z = 8$, and $d = 8$, we shall have $x = 1$, as before.

If we take $y = 4$, $z = 12$, and $p = 10$, then $x = 3$, &c.

QUESTION V.

Find three square numbers, whose sum shall be equal to a given square number a^2.

Here we have

$$x^2 + y^2 + z^2 = a^2.$$

Put $y^2 = 2xz$, and we have

$$x^2 + 2xz + z^2 = a^2;$$

therefore $x + z = a$, or $x = a - z$; and by substitution,

$$y^2 = 2az - 2z^2,$$

which denote by p^2z^2, and we obtain

$$2a - 2z = p^2z, \text{ whence } z = \frac{2a}{p^2 + 2};$$

$$\text{therefore } x = a - \frac{2a}{p^2 + 2}.$$

If we take $a = 9$, and $p = 4$, then $z = 1$, $x = 8$, and $y = 4$.

Hence the three squares are 1, 16, and 64: and

$$1 + 16 + 64 = 9^2.$$

QUESTION VI.

Find four numbers such, that if their sum be multiplied by any one increased by unity, the products shall all be squares.

Let $w^2 - 1$, $x^2 - 1$, $y^2 - 1$, and $z^2 - 1$, be the four numbers; then all the conditions will be fulfilled, if we make

$$w^2 + x^2 + y^2 + z^2 - 4 = \square;$$

in order to which, put $w^2 = 4$, then there only remains to make $x^2 + y^2 + z^2 = \square$, which has been already done, Quest. 4, and x, y, and z, may be 3, 4, and 12, respectively: hence the required numbers are 3, 8, 15, and 143.

QUESTION VII.

It is required to divide a number that is equal to the sum of two known squares, a^2 and b^2, into two other square numbers.

Let x^2 and y^2 represent the required squares; then

$$a^2 + b^2 = x^2 + y^2,$$

$$\text{or } a^2 - y^2 = x^2 - b^2;$$

$$\text{that is, } (a + y)(a - y) = (x + b)(x - b).$$

Put $a + y = p(x - b)$, then $a - y = \dfrac{x + b}{p}$, whence

$$y = p(x - b) - a = a - \frac{x + b}{p}, \text{ from which we get}$$

$$x = \frac{bp^2 + 2ap - b}{p^2 + 1}.$$

Suppose $a = 2$, and $b = 9$, and assume $p = 2$, then we have

$$x = 7, \text{ and } y = p(x - b) - a = -6,$$

so that in this case the two required squares are 49 and 36.

QUESTION VIII.

Find three square numbers in arithmetical progression.

Let x^2, y^2, and z^2, represent the three required squares,

$$\text{then } x^2 + z^2 = 2y^2, \text{ and}$$

$$2x^2 + 2z^2 = 4y^2 = \square.$$

Put $x = m + n$, and $z = m - n$, and we have

$$4m^2 + 4n^2 = 4y^2, \text{ or } m^2 + n^2 = y^2:$$

now this last condition is fulfilled by making

$$m = p^2 - q^2, \text{ and } n = 2pq;*$$

therefore, substituting these values of m and n in the above expressions for x and z, we have

$$x = p^2 - q^2 + 2pq,$$
$$z = p^2 - q^2 - 2pq,$$
$$y = p^2 + q^2,$$

p and q being any numbers whatever.

If we take $p = 2$, and $q = 1$, we shall have

$$x = 7, y = 5, \text{ and } z = 1;$$

that is, the three squares will be 7^2, 5^2, and 1^2.

QUESTION IX.

Find four numbers such, that their sum shall be a square; also, if their sum be multiplied by any one of them, and the product be increased by unity, the results shall be all squares.

Let $x - 1$, $x + 1$, $x - y$, and $x + y$, represent the four numbers; then we have to make

$$4x = \square,$$
$$4x^2 - 4x + 1 = \square,$$
$$4x^2 + 4x + 1 = \square,$$
$$4x^2 + 4xy + 1 = \square,$$
$$4x^2 + 4xy + 1 = \square:$$

now the second and third of these expressions are already squares. It only remains, therefore, to make the other three squares. Assume $x = 4$, then the first expression becomes a square, and the fourth and fifth become $65 - 16y$, and $65 + 16y$; put the first of these $= m^2$, and we get

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* It is obvious that $(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$, also

$$(p^2 + q^2)^2 - (2pq)^2 = (p^2 - q^2)^2, \text{ or}$$
$$(p^2 + q^2)^2 - (p^2 - q^2)^2 = (2pq)^2;$$

hence two square numbers may always be readily found such, that their sum or their difference shall be a square.

$y = \dfrac{65 - m^2}{16}$; put the second $= n^2$, and we get

$$y = \frac{n^2 - 65}{16};$$

whence $65 - m^2 = n^2 - 65$, or

$n^2 = 130 - m^2$, which evidently obtains when $m = 3$, when we have $n = 11$; therefore $y = 3\frac{1}{2}$, and, consequently, the three numbers are 3, 5, $\frac{1}{2}$, and $7\frac{1}{2}$.

QUESTION X.

Find three cube numbers, whose sum shall be a cube.

Let x^3, y^3, and z^3, represent the three cubes, and put their sum $= (x + z)^3 = x^3 + 3x^2z + 3xz^2 + z^3$, and there results

$$y^3 = 3x^2z + 3xz^2.$$

Put now $x = pz$, and then

$$y^3 = 3p^2z^3 + 3pz^3,$$

whence $3p^2 + 3p =$ a cube;

therefore we have to find, by trial, a satisfactory value of p, which presents itself in the case $p = \frac{1}{8}$, consequently, if we make $z = 8$, we get $x = pz = 1$; whence $y = 6$, and the three cubes are 1^3, 6^3, and 8^3, whose sum is 9^3: and by making $z =$ any multiple of 8, we may obtain as many integral solutions as we please.

QUESTION XI.

Find three numbers in arithmetical progression such, that the sum of their cubes may be a cube.

Let $a - x$, a, and $a + x$, represent the three required numbers; then the sum of their cubes is $3a^3 + 6ax^2$, which must be a cube; or putting $x = \dfrac{a}{p}$, we have $3a^3 + 6\dfrac{a^3}{p^2} =$ a cube, therefore $3 + \dfrac{6}{p^2} =$ a cube; and if we now put $p^2 = 2n^3$, this last expression will become

$$\frac{3n^3 + 3}{n^3};\ \text{whence } 3n^3 + 3 = \text{a cube},$$

therefore it remains to satisfy the following conditions, viz.

$$2n^3 = p^2,$$
$$3n^3 + 3 = \text{a cube};$$

the first is readily effected by assuming $p = 2nq$, or $2n^3 = 4n^2q^2$, which gives $n = 2q^2$; and, by substitution, the second becomes

$$24q^6 + 3 = \text{a cube},$$

in which a satisfactory value of q immediately presents itself, viz. $q = 1$, which value gives $n = 2$, and $p = 4$; therefore, assuming $a = 4$, we have $x = 1$, and the three required numbers are 3, 4, and 5, which give $3^3 + 4^3 + 5^3 = 6^3$.

If we take $a = 8$, then $x = 2$, and the numbers are 6, 8, and 10, which give $6^3+8^3+\overline{10}|^3 = \overline{12}|^3$, and taking a any other multiple of 4, we may obtain as many integral solutions as we please.

QUESTION XII.

Find three square numbers in arithmetical progression such, that if the root of each be increased by 2, the three sums may be all squares, of which the sum of the first and third shall be also a square.

By Question 8, the general expressions for the roots of three squares in arithmetical progression are

$$p^2 + 2pq - q^2,$$
$$p^2 + q^2$$
$$p^2 - 2pq - q^2,$$

or by taking $q = 1$, these expressions become

$$p^2 + 2p - 1,$$
$$p^2 + 1,$$
$$p^2 - 2p - 1;$$

and adding 2 to each of these, according to the question, we have

$$p^2 + 2p + 1 = \square,$$
$$p^2 + 3 \quad = \square,$$
$$p^2 - 2p + 1 = \square;$$

also, adding the first and third of these expressions together,

$$2p^2 + 2 = \square\,;$$

therefore, since the first and third expressions are already squares, it only remains to make

$$p^2 + 3 = \square,$$
$$2p^2 + 2 = \square,$$

which they will evidently be when $p = 1$; put then $p = m+1$, and we have to make

$$m^2 + 2m + 4 = \square,$$
$$2m^2 + 4m + 4 = \square.$$

In order to effect this, assume the second expression

$$= (nm + 2)^2 = n^2m^2 + 4nm + 4,$$

and there results

$$2m^2 + 4m = n^2m^2 + 4nm,$$

from which we obtain $m = \dfrac{4\,(1-n)}{n^2-2}$; and by substituting this value of m in the first expression, we shall have

$$16\left(\frac{1-n}{n^2-2}\right)^2 + 8\left(\frac{1-n}{n^2-2}\right) + 4 = \square,$$

or multiplying by $\dfrac{(n^2-2)^2}{4}$, and adding together the like terms, we have

$$n^4 - 2n^3 + 2n^2 - 4n + 4 = \square\,;$$

assume this expression

$$= (n^2 - n + \tfrac{1}{2})^2 = n^4 - 2n^3 + 2n^2 - n + \tfrac{1}{4},$$

and we shall then have

$$-4n + 4 = -n + \tfrac{1}{4}, \therefore\ n = \tfrac{5}{4},$$

consequently,

$$m = \frac{4\,(1-n)}{n^2-2} = \tfrac{16}{7}, \text{ and } p = m + 1 = \tfrac{23}{7};$$

therefore the three required squares are $(\frac{303}{49})^2$, $(\frac{578}{49})^2$, and $(\frac{768}{49})^2$, which are in arithmetical progression, the common difference being $\dfrac{309120}{49^2}$; and if we increase the root of each by 2, we shall have

the three squares $(\frac{30}{7})^2$, $(\frac{26}{7})^2$, and $(\frac{16}{7})^2$, of which the sum of the first and third is the square $(\frac{34}{7})^2$.

13. Find two numbers x and y such, that their sum and difference shall both be squares.

Ans. 4 and 5.

14. Find two square numbers such, that if each be increased by the root of the other, the sums shall both be squares.

Ans. $\frac{1}{36}$ and $\frac{1}{144}$.

15. Find two numbers such, that if the square of each be added to their product, the sums shall both be squares.

Ans. 9 and 16.

16. Find two fractions such, that if either of them be added to the square of the other, the sums may be equal, and that the sum of their squares may be a square number.

Ans. $\frac{3}{7}$ and $\frac{4}{7}$.

17. Find two numbers such, that if their product be added to the sum of their squares, the result may be a square.

Ans. 3 and 5.

18. Find three numbers such, that if to the square of each the product of the other two be added, the results shall all be squares.

Ans. 9, 73, and 328.

19. Find two numbers such, that their sum, the sum of their squares, and the sum of their cubes, may all be squares.

Ans. 184 and 345.

20. Find three numbers such, that their product increased by unity shall be a square, also the product of any two increased by unity shall be a square.

Ans. 1, 3, and 8.

21. Find three numbers, whose sum shall be a square, such, that if the square of the first be added to the second, the square of the second to the third, and the square of the third to the first, the sums shall be all squares.

Ans. $\frac{3}{80}$, $\frac{1}{40}$, and $\frac{1}{20}$.

22. Find three numbers in arithmetical progression such, that the sum of every two may be a square.

Ans. $120\frac{1}{2}$, $840\frac{1}{2}$, and $1560\frac{1}{2}$.

23. Find three numbers in geometrical progression such, that the difference of every two may be a square.

Ans. 567, 1008, and 1792.

24. Find three square numbers such, that the sum of every two may be a square.

Ans. $\overline{44}|^2$, $\overline{117}|^2$, and $\overline{240}|^2$.

25. Find three square numbers such, that the difference of every two may be a square.

Ans. $\overline{153}|^2$, $\overline{185}|^2$, and $\overline{697}|^2$.

26. Find three square numbers in geometrical progression such, that if any one of them be increased by its root, the sum shall be a square.

Ans. $(\frac{49}{480})^2$, $(\frac{98}{480})^2$, and $(\frac{196}{480})^2$.

27. Find three square numbers that shall be in harmonical proportion.

Ans. 1225, 49, and 25.

28. Find two numbers such, that their sum shall be equal to the sum of their cubes.

Ans. $\frac{5}{7}$ and $\frac{8}{7}$

29. Find three cubes such, that if unity be subtracted from each, the sum of the remainders shall be a square.

Ans. $(\frac{17}{15})^3$, $(\frac{11}{15})^3$, and 2^3.

30. Find two numbers such, that their sum shall be a square, their difference a cube, and the sum of their squares a cube.

Ans. 23958 and 34606.

31. Find four numbers such, that the product of any three increased by unity shall be a square.

Ans. $\frac{1}{2}$, 2, 3, and $\frac{16806}{621}$.

NOTE B.—(*Page* 190.)

The amount of £1 for one year, increasing at compound interest, due at every xth part of a year, is, as in the text,

$$A = a(1 + \frac{r}{x})^x,$$

which is calculable, even when x is infinitely great, as we have already shown. It may be inquired, however, whether A in these circumstances be the greatest possible or not. This may be ascertained as follows.

By the binomial theorem,

$$A = a(1+\frac{r}{x})^x = a\{1 + r + \frac{x(x-1}{2} \cdot \frac{r^2}{x^2} + \frac{x(x-1)(x-2)}{2 \cdot 3} \cdot \frac{r^3}{x^3} + \&c.\};$$

and since, for any finite value of x, $x(x-1)$ is less than x^2; $x(x-1)(x-2)$ less than x^3, &c. it follows that

$$a(1 + \frac{r}{x})^x < a\{1 + r + \frac{r^2}{2} + \frac{r^3}{2 \cdot 3} + \frac{r^4}{2 \cdot 3 \cdot 4} + \&c.\}.$$

But when x is infinite, then the proposed series becomes

$$a(1 + \frac{r}{\infty})^\infty = a\{1 + r + \frac{r^2}{2} + \frac{r^3}{2 \cdot 3} + \frac{r^4}{2 \cdot 3 \cdot 4} + \&c.\}.$$

Hence $x = \infty$ makes A a maximum, and equal to this last series, which is the development of $a \times e^r$ (see page 172); hence, as in the text,

$$\log. A = \log. a + r.$$

The series just given is remarkable, showing that

$$(1 + \frac{1}{\infty})^\infty = e = 2{\cdot}718281828 \text{ (see page 181.)}$$

SUPPLEMENT,

CONTAINING

THE PROPERTIES AND SOLUTIONS OF EQUATIONS.

[FROM THE FIRST LONDON EDITION.]

CHAPTER I.

ON THE GENERATION AND PROPERTIES OF EQUATIONS IN GENERAL.

(1.) THE root of an equation is any number, or quantity, which, when substituted for the unknown in that equation, satisfies all its conditions. If, for example, in the general equation,

$$x^n \pm Px^{n-1} \pm Qx^{n-2} \ldots \pm Tx \pm U = 0^*,$$

two, three, four, &c. numbers, or quantities can be found, which, when substituted in the place of x, will make the left-hand side $= 0$, then the equation is said to have two, three, four, &c. roots.

PROPOSITION I.

If any two numbers be raised to the same power, the difference of those powers is divisible by the difference of the two numbers.

Let x and y be any two numbers, then will $x^n - y^n$ be divisible by $x - y$.

* In investigating the properties of equations in the present chapter, to avoid confusion, the double sign $\pm$ has not been preserved: but the generality of the several conclusions is not at all affected by our taking the several terms of the equation with the single sign $+$, since any of the coefficients may be taken negatively, as $Q = -q$, &c.

For put $x - y = z$; then $x = z + y$, and

$$\therefore x^n - y^n = (z + y)^n - y^n;$$

and expanding $(z + y)^n$, we have $(z + y)^n - y^n =$

$$z^n + nz^{n-1}y + \frac{n(n-1)}{2}z^{n-2}y^2 + \ldots . nzy^{n-1} + y^n - y^n =$$

$$y(y^{n-1} + ny^{n-2}y + \ldots . ny^{n-1}),$$

which is evidently divisible by y, $\therefore x^n - y^n$ is divisible by $x - y$.

Cor. If the division of $x^n - y^n$ by $x - y$ be actually performed, the quotient will be

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \ldots . xy^{n-2} + y^{n-1}.$$

PROPOSITION II.

If the root of any equation, as

$$x^n + \text{P}x^{n-1} + \text{Q}x^{n-2} \ldots . + \text{T}x + \text{U} = 0$$

be represented by a, then the first side of this equation is divisible by the binomial $x - a$.

For, since a is a root of the equation, if it be substituted in the place of x, all the conditions will be satisfied; that is, we shall have

$$a^n + \text{P}a^{n-1} + \text{Q}a^{n-2} \ldots . + \text{T}a + \text{U} = 0;$$

and, consequently

$$\text{U} = -a^n - \text{P}a^{n-1} - \text{Q}a^{n-2} \ldots - \text{T}a,$$

so that the proposed equation is the same as

$$\left.\begin{array}{l} x^n + \text{P}x^{n-1} + \text{Q}x^{n-2} \ldots . + \text{T}x \\ -a^n - \text{P}a^{n-1} - \text{Q}a^{n-2} \ldots . - \text{T}a \end{array}\right\} = 0, \text{ or}$$

$$x^n - a^n + \text{P}(x^{n-1} - a^{n-1}) + \text{Q}(x^{n-2} - a^{n-2}) + \text{T}(x - a) = 0:$$

Now the quantities $x^n - a^n$, $x^{n-1} - a^{n-1}$, $x^{n-2} - a^{n-2}$, ... $x - a$, are each divisible by $x - a$ (prop. 1); therefore, the first side of the proposed equation is divisible by $x - a$.

PROPOSITION III.

Every equation has as many roots as there are units in the number denoting its degree; that is, an equation of the nth degree has n roots.

Let there be

$$x^n + \text{P}x^{n-1} + \text{Q}x^{n-2} \dots + \text{T}x + \text{U} = 0;$$

then, if a be one root of this equation, it is the same as

$$x^n - a^n + \text{P}(x^{n-1} - a^{n-1}) + \text{Q}(x^{n-2} - a^{n-2}) \dots + \text{T}(x - a) = 0;$$

and dividing each term by $x - a$, we have

$$\left.\begin{array}{rrrr} x^{n-1} + ax^{n-2} + & a^2x^{n-3} & \dots + & a^{n-1} \\ + \text{P}x^{n-2} + & \text{P}a^2x^{n-3} & \dots + & \text{P}a^{n-2} \\ + & \text{Q}x^{n-3} & \dots + & \text{Q}a^{n-3} \\ & & & \dots\dots \\ & & & + \text{T} \end{array}\right\} = 0, \text{ or}$$

$$x^{n-1} + \text{P}'x^{n-2} + \text{Q}'x^{n-3} \dots\dots \text{T}' = 0,$$

$\text{P}', \text{Q}', \dots \text{T}'$ being put for the sums of the coefficients, or known quantities, under which they are respectively placed.

Now this equation, which is of a degree a unit lower than the proposed equation, must have a root; this is, x must have some value; let then b represent that value, and the equation will be divisible by $x - b$, and the quotient will be of the form $x^{n-2} + \text{P}'\,x^{n-3} + \dots \text{T}''$; hence we have another equation of a degree still lower by a unit, and as x must here also have some value, as c, this equation must be divisible by $x - c$, and if the division be performed, we shall have an equation of a degree still lower by a unit, and, consequently, by proceeding in this manner, we shall at length arrive, after n divisions, to an equation whose degree is $= 0$; therefore, the equation proposed is composed of n factors, viz. $x - a$, $x - b$, $x - c$, &c.; that is $(x - a)(x - b)(x - c) \dots (x - l) = 0$, which equation is verified by making either of the n factors equal to 0; that is to say, by making either $x = a$, or $x = b$, or $x = c$, &c., so that there are n quantities, which, when substituted for x, will satisfy the conditions of the equation, or, in other words, the equation has n roots, a, b, c, d, &c.

SCHOLIUM.

It does not follow from this that all the roots must be different; for any number, or, indeed, all of them may be equal, but still their number is n, since the equation is composed of n factors, and each factor contains a root.

PROPOSITION IV.

No equation has a greater number of roots than there are units in the number denoting its degree.

For, if it be possible, let the equation

$$x^n + Px^{n-1} + Qx^{n-2} + \dots . Tx + U = 0,$$

besides the n roots a, b, c, d, &c. have another root α, not identical with either of the roots a, b, c, &c.; then, since the first side of the equation is divisible by $x - \alpha$ (prop. 2), we have

$$x^n + Px^{n-1} + \text{\&c.} = (x - \alpha)(x^{n-1} + P'x^{n-2} + \text{\&c.}),$$

$$(x - a)(x - b)(x - c) \dots . (x - l) = (x - \alpha)(x^{n-1} + P'x^{n-2} + \text{\&c.}):$$

and since α is a value of x, we have, by substitution,

$$(\alpha - a)(\alpha - b)(\alpha - c) \dots . (\alpha - l) = (\alpha - \alpha)(x^{n-1} + P'x^{n-2} + \text{\&c.}$$

Now the second side of this equation is $= 0$, because $(\alpha - \alpha) = 0$; but the other side cannot be 0, since α is not equal to any of the quantities a, b, c, &c.; hence the supposition is absurd.

Cor. 1. Hence every equation is composed of as many simple equations as there are units in the number denoting its degree, and no more.

Cor. 2. And if one root of an equation be found, and the equation be divided by the simple equation containing that root, the quotient will be an equation containing the other roots; or if any number of roots be found, and the equation be divided by the product of the simple equations containing those roots, the quotient will be an equation containing the remaining roots.

EXAMPLES.

1. One root of the cubic equation $x^3 - 7x^2 + 36 = 0$ is found to be 3; required the other two roots.

Since 3 is one of the roots, the equation is divisible by $x - 3 = 0$; thus,

$$
\begin{array}{l}
x-3)x^3-7x^2+36(x^2-4x-12 \\
\quad\quad x^3-3x^2 \\
\hline
\quad\quad\quad -4x^2+36 \\
\quad\quad\quad -4x^2+12x \\
\hline
\quad\quad\quad\quad\quad -12x+36 \\
\quad\quad\quad\quad\quad -12x+36 \\
\hline
\quad\quad\quad\quad\quad * \quad\quad *
\end{array}
$$

therefore the quadratic $x^2-4x-12=0$, contains the other two roots of the proposed equation, and this quadratic being solved, its roots are found to be 6 and -2, hence the three roots of the cubic are 3, 6, and -2.

2. Given one root of the cubic equation $x^3+x^2-16x+20=0$, equal to -5; required the other two roots.

Ans. 2 and 2.

3. Two roots of the biquadratic equation

$$x^4-3x^3-14x^2+48x-32=0,$$

are 1 and 2; required the other two roots.

Ans. 4 and -4.

Cor. 3. Since the signs in the product $(x+a)(x+b)(x+c)$, &c. are all positive, it follows that if all the roots in any equation be negative, then every term in the equation must be positive, and since in the product $(x-a)(x-b)(x-c)$, &c. the terms are alternately positive and negative; it follows that if all the roots of any equation be positive, the terms of the equation must be alternately positive and negative; and therefore, if the terms in any equation are neither all positive, nor yet alternately positive and negative, that equation must contain both positive and negative roots.*

* It is also equally true, that every equation whose roots are possible, has as many changes of signs from + to —, or from — to +, as there are positive roots, and as many continuations of the same sign from + to +, or from — to —, as there are negative roots. In the simple equation $x+a=0$, there is one continuation, and one negative root, viz. $x=-a$; and in the simple equation $x-b=0$, there is one change, and one positive

PROPOSITION V.

The coefficient of the second term in any equation is equal to the sum of the roots with their signs changed; the coefficient of the third term is equal to the sum of the products of every two roots with their signs changed; the coefficient of the fourth term is equal to the sum of the products of every three roots with their signs changed, &c.; and the last term is equal to the product of all the roots with their signs changed.

Let

$$(x-a)(x-b)(x-c), \text{ \&c.}$$
$$= x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \text{\&c.} = 0;$$

then,

$$(x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \text{\&c.}$$
$$= (x-a)(x^{n-1} + P'x^{n-2} + Q'x^{n-3} + R'x^{n-4} + \text{\&c.})$$
$$\left.\begin{matrix} = x^n + P'x^{n-1} + Q'x^{n-2} + R'x^{n-3} + \text{\&c.} \\ - ax^{n-1} - aP'x^{n-2} - aQ'x^{n-3} - \text{\&c.} \end{matrix}\right\} = 0;$$

$\therefore$ (Art. 2, chap. 4), $P = P' - a$, $Q = Q' - aP'$, $R = R' - aQ'$, &c. Therefore, by introducing the new root a into the equation

$$x^{n-1} + P'x^{n-2} + Q'x^{n-3} + R'x^{n-4} + \text{\&c.} = 0,$$

which may be any equation whatever, the coefficient of the second term becomes increased by $-a$, consequently, if the new root b be introduced into the simple equation $x-a$, the coefficient of the second term in the resulting equation will be $-a-b$; and if c be

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root, viz. $x = b$; also, if these two simple equations be multiplied together, the resulting quadratic $x^2 + \begin{array}{l|} a \\ -b \end{array} x - ab = 0$, will have its second term positive or negative, according as a is greater or less than b; but, in either case, there will evidently be one continuation, and one change of signs, since the equation will be either of the form $x^2 + px - q = 0$, or $x^2 - px - q = 0$, and if into each of these forms either a positive or a negative root be introduced, the resulting equations will be found to have, in like manner, as many changes of signs as positive roots, and as many continuations as negative roots; and if this process be continued, we shall find the rule to hold good in every succeeding equation. This rule is called the rule of Des Cartes.

introduced into this equation, the coefficient of the second term will be $-a-b-c$, &c.; also, the coefficient of the third term in the first case will be $0+ab$, since $Q'=0$; the coefficient of the third term in the succeeding equation will be $+ab-c(-a-b)=ab+ac+bc$, &c.; likewise the coefficient of the fourth term in this equation will be $0-abc$, &c.; so that in the proposed equation it appears that $P=-a-b-c-$ &c.; $Q=ab+ac+bc$, &c.; $R=-abc-$ &c., and so on as announced above.

Cor. 1. Hence, if the coefficient of the second term in any equation be 0; that is, if the term be absent, the sum of the roots will be $=0$, and therefore the sum of the positive roots must be equal to the sum of the negative roots when they are possible.

Cor. 2. Every root of an equation is a divisor of the last term.

PROPOSITION VI.

If the signs of the alternate terms, commencing at the second, be changed, the signs of all the roots will be changed.

For, if the signs of the alternate terms in the equation $x^n+Px^{n-1}+Qx^{n-2}+$ &c. $=0$ be changed, the equation will be $x^n-Px^{n-1}+Qx^{n-2}-$ &c. $=0$, where, if n be even, x must be negative, since the signs of the odd powers are negative; and if n be odd, let the sides of the equation be transposed, then $0=-x^n+Px^{n-1}-Qx^{n-2}+$ &c., where the odd powers of x are negative as before; therefore the values of x, in this and in the preceding equation, are the same as those in the original equation, but with the signs changed.

PROPOSITION VII.

In any equation, whose second term is negative, and all the other terms positive, the coefficient of the second term taken positively, is greater than the greatest root of the equation.

In the general equation $x^n+Px^{n-1}+Qx^{n-2}+$ &c. $=0$, let $P=-p$; then, since $x^n-px^{n-1}=(x-p)x^{n-1}$, the equation may be written thus, $(x-p)x^{n-1}+Qx^{n-2}+$ &c. $=0$; and if p be substituted for x, the first term $(x-p)x^{n-1}$ vanishes, and the other terms being positive, the result is positive; also, if any other quantity greater than p be substituted, the first term, as well as all the others, will be positive.

PROPOSITION VIII.

The greatest negative coefficient of any equation increased by unity is greater than the greatest root of the equation.

Let Q be the greatest negative coefficient $= -q$, and suppose *all* the coefficients are made negative, and equal to q; then our general equation will be

$$x^n - qx^{n-1} - qx^{n-2} - qx^{n-3} - \&c. = 0, \text{ or}$$

$$x^n - q(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + 1) = x^n - q\left(\frac{x^n - 1}{x - 1}\right) = 0;$$

and substituting $q + 1$ for x,

$(q+1)^n - q\left\{\frac{(q+1)^n - 1}{q}\right\} = (q+1)^n - (q+1)^n + 1 = 1$, a positive quantity: If we substitute for x a quantity greater than $q + 1$, as s, then $s^n - q\left(\frac{s^n - 1}{s - 1}\right)$ is greater than 1; for $s^n - (s^n - 1)$ is $= 1$, and $s^n - 1$ is greater than $q\left(\frac{s^n - 1}{s - 1}\right)$ from the supposition; hence $q + 1$ is such a quantity, that when it, or any quantity greater than it, is substituted in the equation, the result will always be positive, and therefore $q + 1$ exceed the greatest positive root of the equation.

PROPOSITION IX.

If a, b, c, &c. be the roots of an equation, of which a is greater than b, b greater than c, &c.; and if a quantity β greater than a be substituted for x, the result will be positive; and if a quantity γ less than a, but greater than b, be substituted, the result will be negative; if a quantity δ less than b, but greater than c, be substituted, the result will be positive, &c.

For by making these substitutions in the equation

$(x - a)(x - b)(x - c)$ &c. $= 0$, we have successively,

$(\beta - a)(\beta - b)(\beta - c)$ &c. $=$ a positive quantity, because all the factors are positive, whether a, b, c, &c. be positive or not;

$(\gamma - a)(\gamma - b)(\gamma - c)$ &c. $=$ a negative quantity, because the first factor is negative and the others positive;

$(\delta - a)(\delta - b)(\delta - c)$ &c. = a positive quantity, because the two first factors are negative, and the others positive, &c. &c.

Cor. 1. Hence, if two quantities be successively substituted for x in any equation, and give results affected with *different* signs, then there lie between those quantities one, three, five, &c., or some *odd* numbers of roots.

Cor. 2. And if the two quantities, substituted for x, give results affected with the *same* sign, then there lie between those quantities two, four, six, &c., or some *even* numbers of roots, or else none at all.

Cor. 3. If any two successive numbers in the arithmetical scale, 0, 1, 2, 3, &c.; or 0, —1, —2, —3, &c.: or 0, ·1, ·2, ·3, &c. &c. be separately substituted for x, and give results affected with different signs, then one root, at least, must lie between those numbers, and therefore the least of the two numbers* substituted must be the first figure of the root; but if the results have the same signs, whatever substitution be made, then an even number of roots must lie between those two numbers, the substitution of which produces results nearest to 0, and, consequently, the least of these two numbers must be the first figure of each of the roots that lie between them, provided the root be possible.

EXAMPLES.

1. Find the first figure in one of the roots of the equation

$$x^3 + 1{\cdot}5x^2 + {\cdot}3x - 46 = 0.$$

It is here obvious that x must be less than 4, for otherwise, x^3 alone would be 64, and therefore the result would be positive, as also for every value greater than 4; let us then try 3, and there is found to result a negative quantity, viz. —2.6, therefore one root must lie between 3 and 4, and the first figure thereof is 3.

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* That is, abstracting from the signs of the two numbers substituted. If the root lie between two negative numbers, as between $-p$ and $-(p+1)$, then $-p$ must be the first figure of the root, which, abstracting from the signs, is less than $-(p+1)$.

2. Find the first figure of one of the roots of the equation

$$x^4 + 3x^3 + 2x^2 + 6x = 148.$$

Here we have to find two consecutive numbers, the first of which when substituted for x, shall make the first side less than 148, and the second, when substituted, shall make it greater than 148.

If we put 2 for x, there results 60, and if we put 3, there results 198, $\therefore$ the first figure of the root is 2.

3. Find the first figure of one of the roots of the equation

$$x^3 - 17x^2 + 54x = 350.$$

Here the two consecutive numbers between which a root lies are 10 and 20, $\therefore$ the first figure of the root is 1 in the tens' place.

PROPOSITION X.

Every equation has an even number of impossible roots, or else none at all.

For, let the equation contain one impossible root, as $a + \sqrt{-\beta}$, then, if the equation be divided by all the simple equations of which it is composed, except two, the result will be a quadratic; let $a + \sqrt{-\beta}$ be one root of this quadratic, then will $a - \sqrt{-\beta}$ be the other (Art. 27, Ch. 3), so that if one root be imaginary, another must be imaginary also; and, in a similar way, it may be shown that if three, or any odd number, be imaginary, there must also be another.

Cor. 1. An equation of an even degree may have all its roots impossible; but if they are not all impossible, two of them at least are possible.

Cor. 2. If all the roots of an equation be impossible, then whatever numbers be substituted for x in that equation, the results will always be affected with the same sign; for if the results ever gave different signs, a root would lie between two possible numbers (Prop. 9, Cor. 1), and therefore might be approximated to, and could not be impossible.

Cor. 3. Since, in every pair of impossible roots, the sign of the one is $+$, and of the other $-$, their product will always be $+$ (Art.

15, Ch. 3): therefore, if an equation have all its roots impossible, the last term must always be positive.

Cor. 4. Hence every equation of an odd degree has at least one real root of a contrary sign to that of the last term; and every equation of an even degree, whose last term is negative, has at least two real roots with contrary signs.

SCHOLIUM.

To determine the number of impossible roots in any equation is a problem of great difficulty, the solution of which has not yet been satisfactorily accomplished. There are, however, several methods by which impossible roots may be detected in any equation, but they are all too difficult and laborious to be admitted into a treatise like the present.*

ON RECURRING EQUATIONS.

(2.) Recurring equations are those, the terms of which, when taken in a direct order, have the same coefficients as the respective terms taken in an inverted order.

PROPOSITION I.

In a recurring equation, one-half the whole number of its roots will be the reciprocals of the other half.

For, in the general recurring equation,

$$x^n + \text{P}x^{n-1} + \text{Q}x^{n-2} + \ldots + \text{Q}x^2 + \text{P}x + 1, \text{ put } x = \frac{1}{y},$$

and it will become

$$\frac{1}{y^n} + \frac{\text{P}}{y^{n-1}} + \frac{\text{Q}}{y^{n-2}} + \ldots + \frac{\text{Q}}{y^2} + \frac{\text{P}}{y} + 1 = 0,$$

* Some of the least difficult of these methods may be seen in Mr. Bridge's Compendious Treatise on the Theory of Equations, lately published.

and multiplying each side by y^n, we have

$$1 + Py + Qy^2 + \ldots + Qy^{n-2} + Py^{n-1} + y^n = 0;$$

an equation the same as the one proposed, except having y in the place of x, and, consequently, the values of x in the one must be the same as the values of y in the other: let then the values of x be a, b, c, &c.; then the values of y are a, b, c, &c.: but $x = \frac{1}{y}$, therefore the values of x are also $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c.

PROPOSITION II.

In a recurring equation of an odd degree, one root will always be -1, provided the equal coefficients have the same sign, and one root will always be $+1$, if the equal coefficients have different signs.

For, in the recurring equation $x^{2n+1} \pm Px^{2n} \pm Q^{2n-1} \pm \ldots Qx^2 \pm Px \pm 1 = 0$, since the coefficient of an odd power of x corresponds to that of an even power throughout all the terms, if the corresponding coefficients have the same sign, and -1 be substituted for x, these coefficients will then have contrary signs, and will therefore destroy each other: and if the corresponding coefficients have different signs, and $+1$ be substituted for x, they will in like manner destroy each other, so that in each case the result will be 0.

Cor. Hence, since -1, or $+1$, is always one root of a recurring equation of an odd degree, the equation is always divisible by either $x + 1 = 0$, or $x - 1 = 0$, and the quotient must always be a recurring equation* of a degree a unit lower.

PROPOSITION III.

A recurring equation of an even degree may always be reduced to an equation of half that degree.

Let the recurring equation be

$$x^{2n} + Px^{2n-1} + Qx^{2n-2} + \ldots + Qx^2 + Px + 1 = 0,$$

* For the roots of the equation arising from the division of the original equation by $x + 1 = 0$, or $x - 1 = 0$, must be the one-half of them reciprocals of the others.

which, by dividing by x^n, becomes

$$x^n + \mathrm{P}x^{n-1} + \mathrm{Q}x^{n-2} + \ldots \mathrm{Q}\frac{1}{x^{n-2}} + \mathrm{P}\frac{1}{x^{n-1}} + \frac{1}{x^n} = 0,$$

or by bringing the terms with equal coefficients together, it is

$$x^n + \frac{1}{x^n} + \mathrm{P}(x^{n-1} + \frac{1}{x^{n-1}}) + \mathrm{Q}(x^{n-2} + \frac{1}{x^{n-2}}) + \&c. = 0;$$

suppose now $2n$ to be successively 2, 4, 6, &c.; then putting $x + \frac{1}{x} = z$, we have the equations

$$x + \frac{1}{x} + \mathrm{P} \ldots\ldots\ldots\ldots = z + \mathrm{P} \ldots\ldots = 0;$$

$$(x^2 + \frac{1}{x^2}) + \mathrm{P}(x + \frac{1}{x}) + \mathrm{Q} \ldots = z^2 - 2 + \mathrm{P}z + \mathrm{Q} = 0;$$

$$(x^3 + \frac{1}{x^3}) + \mathrm{P}(x^2 + \frac{1}{x^2}) + \mathrm{Q}(x + \frac{1}{x}) + \mathrm{R} \ldots\ldots$$

$$\ldots\ldots = (z^3 - 3z) + \mathrm{P}(z^2 - 2) + \mathrm{Q}z + \mathrm{R} = 0;$$

&c. &c.

where the exponent of the highest power of z is successively 1, 2, 3, &c.

Cor. Hence, a recurring equation of the $2n + 1$th degree may be reduced to an equation of the nth degree, since, by the corollary to last proposition, any recurring equation of an odd degree may be depressed a degree lower.

Suppose it were required to find the five roots of the equation $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0$.

Dividing this equation by $x - 1 = 0$, there results

$$x^4 - 12x^3 + 29x^2 - 12x + 1 = 0,$$

where $\mathrm{P} = -12$, and $\mathrm{Q} = 29$,

$$\therefore z^2 + \mathrm{P}z + \mathrm{Q} - 2 = z^2 - 12z + 27 = 0,$$

and, by solving this quadratic, we find the values of z to be 9, and 3; taking the first of those values, we have $x + \frac{1}{x} = 9$, or $x^2 - 9x = -1$, from which equation we get $x = \frac{9}{2} \pm \frac{1}{2}\sqrt{77}$; and taking the second value, we have $x^2 - 3x = -1$, from which $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.

Therefore the five roots of the proposed equation are

$$-1,\ \frac{9+\sqrt{77}}{2},\ \frac{9-\sqrt{77}}{2},\ \frac{3+\sqrt{5}}{2},\ \text{and}\ \frac{3-\sqrt{5}}{2};$$

or if the numerator and denominator of the second of these fractions be each multiplied by $9+\sqrt{77}$, and the numerator and denominator of the last fraction be multiplied by $3+\sqrt{5}$, the roots will assume the following form, viz.

$$-1,\ \frac{9+\sqrt{77}}{2},\ \frac{2}{9+\sqrt{77}},\ \frac{3+\sqrt{5}}{2},\ \text{and}\ \frac{2}{3+\sqrt{5}}.$$

ON BINOMIAL EQUATIONS.

(3.) Binomial equations are those of the form $y^n \pm a^n = 0$; in which, if ax be substituted for y, the form is $a^n x^n \pm a^n = 0$, or dividing by a^n, it is $x^n \pm 1 = 0$, in which form we shall here consider them.

(4.) The following properties of these equations are evident, viz.

1. If n be even, the equation $x^n + 1 = 0$, or $x^n = -1$, has no real root, for $\sqrt[n]{-1}$ is then impossible, therefore its roots are all imaginary.

2. If n be odd, the equation $x^n + 1 = 0$, or $x^n = -1$, has one real root, and no more, for then $\sqrt[n]{-1} = -1$; in this case, therefore, the equation has $n-1$ imaginary roots.

3. If n be even, the equation $x^n - 1 = 0$, or $x^n = 1$, has two real roots, and no more, for $\sqrt[n]{1} = +1$, or -1; therefore it has $n-2$ imaginary roots.

4. If n be odd, the equation $x^n - 1 = 0$, or $x^n = 1$, has only one real root, for then $\sqrt[n]{1} = +1$ only; therefore, in this case, the equation has $n-1$ imaginary roots.

PROPOSITION I.

If a be one of the imaginary roots of the equation $x^n - 1 = 0$; or of the equation $x^n = 1$, then any power of a will be also an imaginary root.

For, since a is one root of the equation $x^n-1=0$, $a^n=1$, therefore $a^{2n}=1$, $a^{3n}=1$, $a^{4n}=1$, &c.; hence, since a, a^2, a^3, a^4, &c. satisfy the equation, when severally substituted for x, these quantities are roots of the equation.

Cor. It hence appears that the roots of $x^n-1=0$, may be represented under an infinite variety of forms, each term in the following series being a root, viz.

$$1,\ a,\ a^2,\ a^3, \ldots a^{n-1},\ a^n,\ a^{n+1},\ a^{n+2}, \ldots a^{2n},\ a^{2n+1},\ a^{2n+2},\ \&c.^*$$

PROPOSITION II.

When n is a prime number†, the roots of the equation $x^n-1=0$ are all contained in the expressions

$$1,\ a,\ a^2,\ a^3, \ldots a^{n-1}, \text{ or } a^n,\ a^{n+1},\ a^{n+2} \ldots a^{2n-1}, \text{ or}$$
$$a^{2n},\ a^{2n+1},\ a^{2n+2} \ldots a^{3n-1},\ \&c.;$$

for in each of these series of roots, all the n terms will be different.

It has been shown in the last proposition, that each of the quantities $1, a, a^2, a^3, \ldots a^{n-1}$, &c. is a root of the equation, if a is a root; if, therefore, no two of the n quantities announced in this proposition be identical, that is, if no two be the same values under different forms, these quantities will represent the whole n roots of the equation; now if we suppose any two to be equal, as $a=a^4$, then $1=a^3$, which is absurd, since a^3 is imaginary (Art. 4); also, if $a^3=a^5$, there would be the absurdity, $1=a^2$, &c.; therefore, if n be a prime number, the n roots of the equation $x^n-1=0$ will be

$$\begin{array}{ll} 1,\ a,\ a^2,\ a^3 \ldots\ldots a^{n-1}, \\ a^n,\ a^{n+1},\ a^{n+2} \ldots\ldots a^{2n-1}, \\ \&c. \qquad\qquad \&c. \end{array}$$

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* It also follows, that since the sum of the roots of any equation is equal to the coefficient of its second term taken with a contrary sign, and their product equal to its last term taken with a contrary sign, the sum of the roots in the equation x^n-1 is $=0$, and their product is $+1$, therefore the sum of the roots, as also their product, is always the same for every value of n.

† A prime number is that which has no other divisors but itself and unity; that is, it cannot be produced by the multiplication of integral factors. Other numbers are called composite numbers.

PROPOSITION III.

When n is not a prime number, the roots of the equation are not all contained in the series

$$1, a, a^2, a^3 \ldots\ldots a^{n-1}; \text{ or } a^n, a^{n+1}, a^{n+2} \ldots\ldots a^{2n-1}; \text{ or } a^{2n}, a^{2n+1}, a^{2n+2} \ldots\ldots a^{3n-1}, \text{ \&c.}$$

for some of these will be the same roots under different forms.

Since n is a composite number, let it be formed from the two primes p, q, of which q is the greater, then each term in the series $1, a, a^2, a^3 \ldots\ldots a^{n-1}$, or which is the same thing, each term in the series

$$1, a, a^2, a^3, \ldots\ldots a^p, a^{p+1}, a^{p+2} \ldots\ldots a^q, a^{q+1} \ldots\ldots a^{pq-1},$$

is a root of the equation (Prop. 1).

Now, since $a^{pq} = 1$, $a^p = \sqrt[q]{1} = 1$; also, $a^q = \sqrt[p]{1} = 1$; therefore the terms 1, a^p, and a^q, are each equal to 1, and, consequently, each must be the same root under a different form (Art. 4): the above n quantities, therefore, do not contain all the roots of the equation, and the same may be shown of the other series of n quantities. If n be composed of more than two prime factors, then it is obvious that there will be more identical terms.

PROPOSITION IV.

To find the roots of the equation $x^n - 1 = 0$, when n is the square of a prime number p.

Put $x^p = y$; then $y^p = 1$, or $y^p - 1 = 0$.

Let the roots of this last equation be $1, \beta, \beta^2, \beta^3, \ldots \beta^{p-1}$; then by substitution,

$$x^p - y = \begin{cases} x^p - 1 = 0, \\ x^p - \beta = 0, \\ x^p - \beta^2 = 0, \\ x^p - \beta^3 = 0, \\ \text{\&c.} \quad \text{\&c.} \end{cases}$$

hence, the pp values of x in these p equations, will evidently be all different, and will be the roots of the equation $x^{pp} - 1 = 0$.

To determine these roots, put $x = z\sqrt[p]{\beta}$, then

$$x^p - \beta = (z^p - 1) \times \beta = 0;$$

therefore the roots of $x^p - \beta = 0$, are equal to the roots of $x^p - 1 = 0$ multiplied by $\sqrt[p]{\beta}$;

that is, $\sqrt[p]{\beta}$, $\beta\sqrt[p]{\beta}$, $\beta^2\sqrt[p]{\beta}$, &c.

in the same manner we find the roots of $x^p - \beta^2 = 0$, $x^p - \beta^3 = 0$, &c.

therefore the roots of

$$\left.\begin{array}{l} x^p - 1 = 0 \text{ are } 1, \beta, \beta^2, \beta^3 \ldots\ldots \beta^{p-1} \\ x^p - \beta = 0 \ldots \sqrt[p]{\beta}, \beta\sqrt[p]{\beta}, \beta^2\sqrt[p]{\beta} \ldots \beta^{p-1}\sqrt[p]{\beta} \\ x^p - \beta^2 = 0 \ldots \sqrt[p]{\beta^2}, \beta\sqrt[p]{\beta^2}, \beta^2\sqrt[p]{\beta^2} \ldots \beta^{p-1}\sqrt[p]{\beta^2} \\ \quad\&c. \qquad\qquad \&c. \qquad\qquad \&c. \end{array}\right\} \begin{array}{l} = \text{the } n \text{ roots} \\ \text{of } x^n - 1 \\ = 0, \end{array}$$

In a similar manner it may be shown that the roots of the equation $x^{ppp} - 1 = 0$, &c. are those of pp, &c. equations of the pth degree.

PROPOSITION V.

To find the roots of the equation $x^n - 1 = 0$, when n is the product of two primes p and q.

Put $x^p = y$; then $y^q - 1 = 0$.

Let the roots of this last equation be

$$1, a, a^2, a^3, \ldots . a^{q-1},$$

$$\text{or } 1, a^p, a^{2p}, a^{3p} \ldots a^{\overline{q-1}\times p} \text{ (Prop. I)}:$$

$$\text{now } x^p - y = \begin{cases} x^p - 1 = 0, \\ x^p - a^p = 0, \\ x^p - a^{2p} = 0, \\ x^p - a^{3p} = 0, \\ \&c. \qquad \&c. \end{cases}$$

And if the values of x in the equation $x^p - 1 = 0$, be denoted by 1, b, $b^2 \ldots b^{p-1}$, then the values of x in the equation $x^p - a^p = 0$, will be a, ab, $ab^2 \ldots ab^{p-1}$; the values of x in the equation $x^p - a^{2p} = 0$, will be a^2, a^2b, $a^2b^2 \ldots a^2b^{p-1}$; and so on. Hence all the roots of the given equation $x^{pq} - 1 = 0$, will be

$$\begin{array}{l} 1,\ b,\ b^2,\ b^3 \ldots\ldots\ldots b^{p-1}, \\ a,\ ab,\ ab^2,\ ab^3 \ldots\ldots ab^{p-1}, \\ a^2,\ a^2b,\ a^2b^2,\ a^2b^3 \ldots\ldots a^2b^{p-1}, \\ a^3,\ a^3b,\ a^3b^2,\ a^3b^3 \ldots\ldots a^3b^{p-1}, \\ \quad\&c. \qquad \&c. \qquad \&c. \end{array}$$

Thus, to find the fifteen roots of the equation $x^{15}-1=0$, or $x^{3\cdot 5}-1=0$:
here $p=3$, and $q=5$;
whence the roots will be

$$\left.\begin{array}{l} 1,\ b,\ b^2, \\ a,\ ab,\ ab^2, \\ a^2,\ a^2b,\ a^2b^2, \\ a^3,\ a^3b,\ a^3b^2, \\ a^4,\ a^4b,\ a^4b^2, \end{array}\right\}\text{roots or values of } x,$$

where the values of b and b^2, are contained in the expression, $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$; and the values of a, a^2, a^3, and a^4, in the expression,

$$\tfrac{1}{4}[(-1\pm\sqrt{5})\pm\sqrt{(-10\mp 2\sqrt{5})}].$$

If n be the product of three primes pqr, put $pq=k$, and then find the roots of the equation, $x^{rk}-1=0$; and so we may proceed for any number of terms.

Cor.—From the above proposition and the preceding one, it follows that the binomial equation $x^n-1=0$, may be solved by means of equations not higher than quadratics, when n is of the form $2^\alpha\times 3^\beta\times 5^\gamma$; and by means of equations not higher than a cubic, when n is of the form $2^\alpha\times 3^\beta\times 5^\gamma\times 7^\delta$; since, if $n=5$, the equation $x^5-1=0$ being divided by $x-1=0$, becomes

$$x^4+x^3+x^2+x+1=0;$$

and if $n=7$, the equation divided by $x-1=0$, becomes

$$x^6+x^5+x^4+x^3+x^2+x+1=0;$$

and both of these being recurring equations, may be solved by equations of half their dimensions, (Art. 2, Prop. iii. Cor.)*

---

* The solution of binomial equations has been shown by M. Gauss, Professor of Mathematics at Gottingen, to be applicable to the *geometrical* division of the circle into any prime number of equal parts of the form 2^m+1; a problem, that till the commencement of the present century, was supposed to be impossible. An ingenious chapter relating to Gauss's celebrated theorem may be seen in Barlow's Theory of Numbers.

CHAPTER II.

ON THE SOLUTION OF CUBIC EQUATIONS, AND THE EXTRACTION OF THE CUBE ROOT.

ON CUBIC EQUATIONS.

(1.) Let $x^3 + \text{C}x^2 + \text{B}x = \text{N}$ be any cubic equation, and suppose that two consecutive numbers in either of the series 1, 2, 3, &c., or 10, 20, 30, &c.; or ·1, ·2, ·3, &c. &c. are found such, that by substituting the first for x in the above equation, the result shall be less than N, and by substituting the second, the result shall be greater than N; then the first of these numbers will be the first figure of one of the roots of the equation (Ch. 5, Prop. 9). Let this first figure be represented by r, and the other succeeding figures of the same root by s, t, u, &c. respectively, then if N be divided by $r^2 + \text{C}r + \text{B}$, the quotient must evidently be r; let the remaining figures of the root s, t, u, &c. be called y, and then $x = r + y$, or $y + r = x$; whence,

$$\begin{array}{rcl} \text{B}y + \text{B}r & = & \text{B}x, \\ \text{C}y^2 + 2\text{C}ry + \text{C}r^2 & = & \text{C}x^2, \\ y^3 + 3ry^2 + 3r^2y + r^3 & = & x^3, \\ \hline y^3 + \text{C}'y^2 + \text{B}'y + \text{A} & = & \text{N}, \end{array}$$

the terms in the last line being the sums of those under which they are respectively placed, C′, B′, A, and N, being known numbers. Now, by transposing A, we have

$y^3 + \text{C}'y^2 + \text{B}'y = \text{N} - \text{A}$, or putting N′ for N — A,

$y^3 + \text{C}'y^2 + \text{B}'y = \text{N}'$, an equation similar to the first; and since s is the first figure of the root y of this equation, if N′ be divided by $s^2 + \text{C}'s + \text{B}'$, the quotient must evidently be s; suppose the value of s found, and let the remaining figures t, u, &c. be called z, then $y = s + z$, or $z + s = y$; whence,

26

$$
\begin{array}{r}
\text{B}'z + \text{B}'s = \text{B}'y, \\
\text{C}'z^2 + 2\text{C}'sz + \text{C}'s^2 = \text{C}'y^2, \\
z^3 + 3sz^2 + 3s^2z + s^3 = y^3, \\
\hline
z^3 + \text{C}'z^2 + \text{B}''z + \text{A}' = \text{N}';
\end{array}
$$

therefore, by transposing A', and putting N'' for $\text{N}' - \text{A}'$, we have $z^3 + \text{C}''z^2 + \text{B}''z = \text{N}''$, an equation also similar to the first, in which C'', B'', and N'', are known numbers. We may now proceed to find the first figure t, in the root z, of this equation, the value of which must be such, that if N'' be divided by $t^2 + \text{C}''t + \text{B}''$, the quotient will be t; and by continuing this process we may find, one by one, all the figures of the root x of the proposed equation.

(2.) Now, from observing the formation of the coefficients C', B', in the second equation, and recollecting that r, being the first figure in the root, must be greater than s, it will appear obvious that B' must form a great part of the divisor $s^2 + \text{C}'s + \text{B}'$; and if r be already known, the value of B' will become known, which may therefore be used as a trial divisor for determining the next figure of the root; the same may be observed of the next and the succeeding divisors; but in these the trial divisors B'', B''', &c. will continually approach nearer to the true divisors.

(3.) If now the first figure r of the root be found by trial, and $r + \text{C}$ be multiplied by it, and the product added to B, we shall have the first divisor: thus,

$$
\begin{array}{l}
\text{B} \\
r^2 + \text{C}r \\
\hline
r^2 + \text{C}r + \text{B} \text{ the first divisor;}
\end{array}
$$

and if underneath the first divisor we place r^2, and add it to the two expressions immediately above, we shall get B': thus,

$$
\begin{array}{l}
\text{B} \\
r^2 + \text{C}r \\
\hline
r^2 + \text{C}r + \text{B} = \text{1st divisor} \\
r^2 \\
\hline
3r^2 + 2\text{C}r + \text{B} = \text{B}';
\end{array}
$$

therefore B′ being obtained, we have a trial divisor of N′ that will enable us to determine, more readily, the next figure s of the root, which, when found, the second divisor may be completed by adding to B′, s times $(s + 3r + \text{C})$, or $s^2 + \text{C}'s$, which will give $s^2 + \text{C}'s + \text{B}'$; and in a similar manner may the succeeding divisors be formed; but the conciseness of the operation will depend, in a great measure, upon the arrangement of the several terms. The process by the following rule will be found to be very simple.

(4.) Put down B, the coefficient of x, and a little to the right place the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient.

Place the first figure of the root, found by trial, in the quotient, above which write the coefficient of x^2, observing that its units' place be over the units' place of the quotient.

Multiply the value of the quotient figure, taking in those above by that value; add the product to B, and the sum is the first divisor.

Write the square of the quotient figure just found under the first divisor, add it to the two sums immediately above, and the result will be the trial divisor for finding the next figure.

Find now the next figure of the root, and to its value, (including those above it,) prefix three times the preceding, taking in the value of the figure above it, multiply the result by the last found figure, add the product to the trial divisor, and we shall have the true divisor; and in the same manner are the succeeding divisors to be obtained.

EXAMPLES.

1. Extract the root of the equation $x^3 + 8x^2 + 6x = 75{\cdot}9$.

Here we find the first figure of the root to be 2, therefore the operation will be as follows:

	N $8 = C$
$B = 6$	75·9 (2·425 the root
$r + C = 10 \ldots 20$	52
$r^2 + Cr + B = 26$	23·9 (N')
$r^2 = 4$	22·304
$3r^2 + 2Cr + B = 50$	1·596 (N'')
14·4 . . . 5·76	1·239688
55·76	·356312
·16	·311827625
61·68	44484375
15·22 . . . ·3044	
61·9844	
4	
62.2892	
15·265 . . . 76325	
62·365525	

In the foregoing operation, which has been performed exactly according to the rule, it will be perceived, that after the first decimal place in the root has been found, more decimals have been used in the succeeding parts of the work than were absolutely necessary for the extent to which the root has been carried; for, if after having obtained so many as three places of decimals in the last column of the work, we had ceased to admit any more, and had rejected all the other places to the right, we should still have had the root equally correct to three places of decimals. Now, in order that the number of decimals in the last column may not exceed three, it is obvious that the divisor corresponding to the first decimal in the root must contain but two decimals; the divisor corresponding to the next decimal of the root must contain but one; and the one corresponding to the next succeeding root figure must not contain any; and that for every succeeding decimal in the root the right hand digit in the cor-

responding divisor must be struck off.* In the same manner, after one decimal in the root is obtained, the numbers in the first column are to be diminished, in order that the decimals in the second column may not exceed the necessary number: It therefore follows that the operation of annexing each new figure of the root to thrice the preceding, as also that of placing the square of each new figure under the preceding divisor, become quite unnecessary after the first decimal in the root has been found. Hence the work of the preceding example may be rendered more concise, and will stand as follows:

```
                          8
                 6       75·9(2·4257 the root
      10 . . . 20        52
               ----      ----
                26       23·9
                 4       22·304
               ----      ------
                50        1·596
    14·4 . . .  5·76      1·240
               -----     ------
                56·76     ·356
                  ·16     312
               ------    -----
                61·68      44
 1|5·|2 . . .     ·30      43
               ------      --
                61·98       1
               -----|       -
                62·3
                   1
               -----
                62·4
                 ||
               -----
```

* It must be observed, however, that although a figure is thus cut off each time, yet, in the multiplication, the product that would have arisen from this figure is to be ascertained; and although nothing is to be put down, yet, what would have been carried, is still to be carried for the increase of the next figure, and, indeed, if the figure that would have been put down, be 5, or upwards, then one *unit more* is to be carried to the next figure, exactly the same as in *contracted multiplication*. Thus, although in the operation in the text the figure 8 is struck off from the divisor 6198, yet, since the product of the 8 by 2 is 16, 2 is carried for the increase of the next figure.

Here the two last figures of the root are obtained by plain division, the figures in the first column having all been struck off.

2. Extract the root of the equation $x^3 + x^2 = 500$, to about 9 places of figures.

Here the first figure of the root is 7.

```
                          1
            0           500(7·61727975 the root
   8 ··· 56             392
         ——             ———
         56             108
         49             104.736
         ——             ———————
        161               3.264
  22.6 .. 13.56           1.887181
        ——————            ————————
        174.56            1.376819
             36           1.323862
        ——————            ————————
        188.48              52957
 23.81 ...  2381            37859
        ————————            —————
        188.7181            15098
               1            13251
        ————————            —————
        188.9563             1847
2|3.8|37 ... 1669            1704
        ————————             ————
        189.123|2             143
        —————————             133
        189.290               ———
              5                10
        ————————                9
        1|8|9.|2|9|5            —
        ————————————
```

3. Extract the root of the equation $x^3 + \frac{11}{16}x = \frac{3}{4}$, to 5 places of figures.

Putting the equation under the form $x^3 + \cdot 6875x = \cdot 75$, we find the first figure of the root to be ·6.

		0
	·6875	·75(·66437 the root
	·36	·62850
	1.0475	·12150
	·36	·11275
	1.7675	875
1.|86 ··	·1116	800
	1.879|1	75
	4	60
	1.994	15
	7	14
	2.|0|0|1	

4. Extract the root of the equation $x^3 - 17x^2 + 54x = 350$ to about ten places of figures.

Here the first denomination of the root is 10.

		−17
	54	350(14.95406861 the root
−7 . . .	−70	−160
	−16	510
	100	328
	14	182
17 . . .	68	170.379
	82	11.621
	16	10.740875
	166	·880125
25.9 . . .	23.31	·865276
	189.31	14849
	.81	12986
	213.43	1863
27.75 . . .	1.3875	1731
	214.8175	132
	25	130
	216.2075	2
27.85 . . .	.1114	2
	216.3189	—
	216.430	

5. Extract the root of the equation $x^3 + 24\cdot84x^2 - 67\cdot613x = 3761\cdot2758$ to about ten places of figures.

Here the highest denomination of the root is 10.

		24.84
	— 67.613	3761.2758(11.19733377 the root
34.84 . . .	348.4	2807.87
	280.787	953.405
	100	785.027
	729.187	168.3788
55.84 . . .	55.84	84.7661
	785.027	83.6127
	1	77.283513
	841.867	6.329187
57.94 . . .	5.794	6.050544
	847.661	·278643
	1	·259437
	853.465	29206
58.23 . . .	5.2407	25944
	858.7057	3262
	81	2594
	863.9545	668
58.41 . . .	·4089	605
	864.3634	63
		60
	864.772	
	17	3
	864.789	
	864.81	

6. Extract the root of the equation $x^3 + 2x^2 + 3x = 13089030$.

Here the highest denomination of the root is 200.

		2
		13089030(235 the root
202 . . .	40400	80806
	40403	500843
	4	419289
	120803	815540
632 . . .	1896	815540
	139763	
	9	
	159623	
697 . . .	3485	
	163108	

7. Extract the root of the equation $x^3 - 7x = -7$.

In this equation, whatever term in the series 0, 1, 2, 3, &c. is substituted for x, the result is always greater than -7, therefore, if its roots are possible, two of them must lie between that pair of numbers in the above series, the substitution of which produces results the nearest to -7 (Prop. 9, Cor. 3, Ch. 5), which are found to be the numbers 1 and 2; therefore 1 must be the first figure of a root.

```
             — 7             — 7(1.356895 the root
               1             — 6
             ———             ———
             — 6             — 1
               1             — ·903
             ———             ——————
             — 4             — ·097
3.3 . . .      ·99            — 86625
             ——————           ————————
             — 3.01           — 10375
                  9            — 9049
             ——————            ———————
             — 1.93            — 1326
3.95 . . .      ·1975          — 1185
             ————————          ———————
             — 1.7315            — 141
                   25            — 133
             ————————            —————
             — 1.5325              — 8
4.05 . . .        243              — 7
             ————————              ———
             — 1.5082              — 1
             ————————              ———
             — 1.484
                   3
             ————————
             — 1.481
```

8. Extract the root of the equation $x^3 - x^2 - 2x + 1 = 0$. Here the value of x will be found to be between 1 and 2.

```
                              — 1
             — 2             — 1(1.8019377 the root
               0             — 2
             ———             ———
             — 2               1
               1               ·992
             ———             ————————
             — 1                8
2.8 . . .  2.24                 4124
           ————                 ————
             1.24               3876
               64               3720
             ————               ————
             4.12                156
4.4 . . . .    44                124
           ————————              ———
           4.124|4                32
           ———————                29
           4.129                  ——
               4                   3
           ———————                 3
           4.133                   —
                                   1
```

9. Extract the root of the equation $x^3 - 18\frac{1}{12}x = -29\frac{53}{108}$. Here the value of x is between 2 and 3.

	−18.083	−29.49074 (2.33333333 the root indicating $2\frac{1}{3}$ or $\frac{7}{3}$
	4	−28.16
	−14.083	−1.32407
	4	−1.2579
	−6.083	−6607407
6.3 . .	1.89	−601629
	−4.193	−591107
	9	−532695
	−2.213	−58412
6.93 . . .	.2079	−52577
	−2.00543	−5835
	9	−5251
	−1.79663	−584
6.|99|3 . . .	2098	−525
	−1.77565	−59
		−53
	−1.75467	−6
	210	−5
	−1.7525|7	
	−1.7505	
	2	
	−1.7503	

SCHOLIUM.

(5.) The preceding method of solving cubic equations, the author conceives to be more simple and concise than any other that has yet been made public. It will have been perceived that in the preceding method, the whole of the operations requisite to be performed are

actually exhibited, and that no portion of the work has been effaced for the purpose of merely reducing the space, instead of the labour, that the operation might require. The shortest method of extracting the roots of equations, which the author has yet seen, is that investigated by Mr. Horner in the Philosophical Transactions for 1819, part 2; but for cubic equations, this is not so concise as the method here given, and is by no means equal to it as it respects the ease and facility with which the several figures of the root are successively obtained.

(6.) Although in the preceding solutions only one root of each equation has been obtained, yet the others may be had with equal facility by finding the first figure in one of them, and operating as before; but the best and shortest way of proceeding to determine the other roots will be this:

Subtract the root found, taking it with a contrary sign from the coefficient of the second term in the proposed equation, and call the remainder a: divide the absolute number, or second side of the proposed equation, by the root found;* and call the quotient b, then will the quadratic $x^2 + ax + b = 0$, contain the other two roots of the equation (Prop. 5, Ch. 5); that is, if r denote the root of the equation $x^3 + bx + c = 0$, then the quadratic will be

$$x^2 + (a + r)\,x - \frac{c}{r} = 0.$$

ON THE EXTRACTION OF THE CUBE ROOT.

(7.) From the preceding method of extracting the roots of cubic equations may be derived a new method of extracting the cube root of numbers, which will be much more easy and concise than the method usually given.

Suppose, for example, it were required to extract the cube root of the number 12326391, or, which is the same thing, to extract the root of the cubic equation

$$x^3 = 12326391.$$

* If each root be required to more than three or four places of figures, this division will be most readily performed by logarithms.

By proceeding according to the method in article 4, the operation will be as follows:

```
                0            12326391(231
                4             8
               ---           -----
                4             4326
                4             4167
               ---            -----
               12              159391
     63 . . .  189             159391
              -----           -------
               1389           -------
                  9
              -----
               1587
    691 . . .   691
              ------
               159391
              ------
```

From inspecting the above operation, it will be obvious that some of the work is superfluous; thus, the first trial divisor 12 might have easily been found at once by multiplying the square of the root figure 2 by 3; also, since the numbers that are placed under the trial divisors to be added thereto always have two figures to the right, when the addition is performed they are written down again; but this repetition would be avoided if these two numbers were placed *at first* a line lower down, and only the other figures placed *immediately under* the trial divisor, but then, in afterwards adding the square of the new figure, these two figures must be repeated *twice* in the addition, so that we have the following

New method to extract the cube root of any given number.

(8.) Divide the given number into periods of three figures each, as in the common method, and find the nearest cube to the first period, subtract it therefrom, and put the root in the quotient; then thrice the square of this root will be the trial divisor for finding the next figure.

Draw a line a little below the trial divisor, multiply the new figure with thrice the preceding prefixed by the new figure, and place the first two figures of the product *below* this line, and to the right of the trial divisor, and the others *above* the line; add them to the trial divisor, and the sum will be the true divisor.

Under this divisor write the square of the last root figure, which add to the two sums above, repeating the two first figures of the divisor twice, and the result is the next trial divisor; the true divisor is found as before, &c.

NOTE. After the first or second decimal place in the root is found, the square of the root figure used in forming the trial divisor may be omitted, as also those two figures that would fall below the line in forming the true divisor, as the value of these figures will be too small for their omission to affect the truth of the result. But if the number of decimals in the root is required to be very great, these omissions must not be made till after the third or fourth decimal in the root is found.*

The preceding example by this rule will stand thus:

	12	12326391(231
	1	8
63 . . .	——	——
	1389	4326
	9	4167
	——	——
	1587	159391
	6	159391
691 . . .	——	——
	159391	——
	——	

* At whatever divisor these contractions take place, as many more decimals of the root will be obtained as there are figures in this divisor *minus* one, although the last decimal thus obtained, if the root has been extended to fourteen or fifteen places, is not always to be depended upon.

2. Extract the cube root of 3 to three places of decimals.

	3	3(1.442
	1	1
3.4 . . .	—	—
	4.36	2
	.16	1.744
	—	—
	5.88	.256
	.17	.242
4.2 . . .	—	—
	605	14
	— |	12
	62	—
	|	2
	—	—

3. Extract the cube root of 3 to six places of decimals.

	3	3(1.442249
	1	1
3.4 . . .	—	—
	4.36	2
	.16	1.744
	—	—
	5.88	.256
	.16	.241984
4.24 . . .	—	—
	6.0496	14016
	16	12459
	—	—
	6.2208	1557
	86	1248
4.32 . . .	—	—
	6.2294	309
	— |	250
	6.238	—
	1	59
	—	56
	6.239	—
	— |	3
	624	—
	| |	
	—	

4. Extract the cube root of 9 to about fourteen or fifteen places of decimals.

	12	9(2.080083823051904
		8
6.08 . . .		
	12.4864	1
	64	.998912
	12.9792	1088
	4	1038375936512
6.24008 . . .		
	12.9796992064	49624063488
	64	38940651420
	12.9801984192	10683412068
	187207	10384192682
6.24024 . . .		
	12.9802171399	299219386
		259604919
	12.980235861	
	4992	39614467
		38940738
	12.980240853	
		673729
	12.98024585	649012
	12	
		24717
	12.98024597	12980
		11737
	129802461	11682
		55
		52
		3

CHAPTER III.

ON THE SOLUTION OF EQUATIONS OF THE HIGHER ORDERS.

PRELIMINARY ARTICLE ON BINOMIAL COLUMNS.

(9.) It has been already shown, that in the expansion of a binomial, the coefficient of the last term is always 1, whatever may be the index; the coefficient of the nth term, when the index is n, is n also; the coefficient of the nth term, when the index is $n+1$, is $\frac{n(n+1)}{2}$ &c.; that is,

$$\text{the coefficient of the } n\text{th term, when the index is} \begin{cases} n-1 \text{ is} \;.\;.\;.\; 1 \\ n \;.\;.\;.\;.\;.\;.\; n \\ n+1 \;.\;.\;.\; \frac{n(n+1)}{2}, \\ n+2 \;.\;.\; \frac{n(n+1)(n+2)}{2.3} \\ n+3 \;.\;.\; \frac{n(n+1)(n+2)(n+3)}{2.3.4}, \end{cases}$$

$$\&c. \qquad \&c.$$

Therefore, if the exponent $n-1$ of a binomial be continually increased by 1, and the first, second, third, &c. terms in the successive expansions be placed respectively under each other, the above expressions will represent the coefficients in the nth vertical column, which may be called the nth binomial column: Now let the value of n be successively 1, 2, 3, 4, &c. and we shall then have the coefficients in the first, second, third, fourth, &c. column as follow:

Binomial Columns.

1st	2nd	3rd	4th	5th	6th	&c.
1	1	1	1	1	1	&c.
1	2	3	4	5	6	
1	3	6	10	15	21	
1	4	10	20	35	56	
1	5	15	35	70	126	
1	6	21	56	126	252	
&c.	&c.	&c.	&c.	&c.	&c.	&c.

From having the nth column, the succeeding, or $n+1$th, may be obtained by substituting $n+1$ for n; so that the $n+1$th column is

$$1$$
$$n+1,$$
$$\frac{(n+1)(n+2)}{2},$$
$$\frac{(n+1)(n+2)(n+3)}{2.3},$$
$$\frac{(n+1)(n+2)(n+3)(n+4)}{2.3.4},$$
&c.

THEOREM.

The mth term in any column is equal to the $m-1$th term in the same column, + the mth term in the preceding.

For, in the above general expressions, it is obvious from the law of the terms, that the mth term in the nth column is

$$\frac{n(n+1)(n+2)\ldots(n+m-2)}{2.3\ldots\ldots\ldots m-1)}$$

and the $m-1$th in the $n+1$th column is

$$\frac{(n+1)(n+2)\ldots(n+m-2)}{2.3\ldots\ldots\ldots(m-2)},$$

and the sum of these two is

$$\frac{(n+1)(n+2)(n+3)\ldots(n+m-1)}{2.3\ldots\ldots\ldots\ldots(m-1)},$$

which is evidently the mth term in the $n+1$th column.

Corollary 1. Hence the second term in any column is equal to the first term in that column + the second term in the preceding column; but the first term in every column is the same, being 1; therefore, the second term in any column is equal to the sum of the two first in the preceding, the third is equal to the sum of the three first in the preceding, and the mth equal to the sum of the m first in the preceding.

Cor. 2. Hence also in any column if the two first terms be written down horizontally, and if underneath the second term the first

be placed, and the operation of addition performed, the result will be the two first terms in the succeeding column; and if in like manner this result be placed under the three first numbers in the former column, the first term under the second, as before, and the addition performed, the result will be the three first terms in the succeeding column, &c.; therefore, from having the numbers in the first column, those in the second may be obtained by performing these several additions, and from these last, those of the third column may be obtained in a similar manner, and thence, those of the fourth, &c. Thus,

1st	2nd	1st	3rd	2nd	1st	4th	3rd	2nd	1st		
1	1	1	1	1	1	1	1	1	1	&c.	nos. of the 1st col.*
	1		2	1		3	2	1			
	2	1	3	2	1	4	3	2	1	&c. . . .	2d col.
	1		3	1		6	3	1			
	3	1	6	3	1	10	6	3	1	&c. . . .	3d col.
	1		4	1		10	4	1			
	4	1	10	4	1	20	10	4	1	&c. . . .	4th col.
	&c.			&c.			&c.			&c.	&c.

Cor. 3. Hence, in any series of the form R, R + Q, R + Q + P, R + Q + P + O, &c.; if the first term be added to the second, the sum to the third, this last sum to the fourth, &c. the coefficients of R, Q; of R, Q, P; of R, Q, P, O, &c.; in these sums will be the numbers of the second binomial column; and if similar operations be performed on the same first term and these sums, the resulting coefficients in the second sums will be the numbers of the third binomial column; and so on, as is obvious from the last Cor., the coefficients of the terms of the series here proposed being respectively 1, 1 1, 1 1 1, 1 1 1 1, &c.

* In this series of units, the right hand unit in each term is taken as the first number in the first column, and for that reason the numbers in the successive columns are produced in a reverse order; if the series of units had been considered as written in a direct order, the other numbers would have been produced in direct order, as is evident; but the former way has been adopted, because we shall have to refer to them in this form hereafter.

Cor. 4. In the same manner, if the series be of the form R, $Rr + Q$, $Rr^2 + Qr + P$, $Rr^3 + Qr^2 + Pr + O$, &c. and the first term be multiplied by r and added to the second, and the sum multiplied by r and added to the third, &c. the numeral coëfficients produced will be the numbers of the second binomial column; and if to the first of these sums r times the first term be again added, and the result be in like manner multiplied by r and added to the second sum, &c. the resulting coefficients will be the numbers of the third binomial column, &c.

HIGHER EQUATIONS RESOLVED.

(10.) Let $Rx^n + \ldots\ldots Ex^4 + Dx^3 + Cx^2 + Bx = N$ be an equation of any degree whatever, and let r be such a number in either of the series 0, 1, 2, 3, &c.; or 10, 20, 30, &c.; or &c.; that when it and the next succeeding number are separately substituted for x in the equation, the results shall be the one less, and the other greater than N; then r will be the first figure of one of the roots of the equation; and if N be divided by $Rr^{n-1} + \ldots\ldots Er^3 + Dr^2 + Cr + B$ the quotient must be r. Suppose such a value of r is found, and let y represent the succeeding figures of the root; then $y + r = x$, and therefore

$$\begin{array}{rr}
 & By + Br = Bx, \\
 & Cy^2 + 2Cry + Cr^2 = Cx^2, \\
 & Dy^3 + 3Dry^2 + 3Dr^2y + Dr^3 = Dx^3, \\
 & Ey^4 + 4Ery^3 + 6Er^2y^2 + 4Er^3y + Er^4 = Ex^4, \\
Ry^n + \ldots\ldots & \\
\hline
Ry^n + \ldots\ldots E'y^4 + D'y^3 + C'y^2 + B'y + A = N; &
\end{array}$$

and by transposition,

$$Ry^n + \ldots\ldots E'y^4 + D'y^3 + C'y^2 + B'y = N - A = N',$$

an equation similar to the one proposed, the first figure s, in the root y of which must be such, that if N' be divided by $Rs^{n-1} + \ldots\ldots E's^3 + D's^2 + C's + B'$, the quotient will be s; if, therefore, we suppose s to be found, and if z be put for the remaining figures of the root, we shall, by proceeding as before, get another equation $Rz^n + \ldots E''z^4 + D''z^3 + C''z^2 + B''z = N''$; also similar to the first; and if we continue this process, we may obtain, one by one, all the figures of the root x, and it is evident that each divisor will be similarly formed from the coefficients of the corresponding equation, and the new figure of the root.

Now it is obvious that $B'y$ is the nth term in the equation $Ry^n + \dots E'y^4 + D'y^3 + C'y^2 + B'y = N'$, or that the column represented by B' is the nth column from the left, and that it consists of n terms; the column represented by C' is the $n-1$th, and consists of $n-1$ terms, &c.; also each of these columns, omitting the numeral parts, is equal to the preceding multiplied by r, *plus* the corresponding coefficient in the proposed equation; consequently, since the first column is simply R, if the second, third, &c. coefficients of the proposed equation be Q, P, &c. respectively, then the first, second, third, &c. columns, without the numeral coefficients, will be R, $Rr + Q$, $Rr^2 + Qr + P$, &c.; but it has been shown (see the preceding article 1, cor. 4), that if the first term in this series be multiplied by r, and the product added to the second, and the result be multiplied by r, and the product added to the third, &c. &c. that the proper numeral coefficients will be obtained, because the above columns of numeral coefficients are the same as the several binomial columns, except that the above are written in a reverse order, that is, from right to left, and in this reverse order they will be produced by adding the above series as directed. Hence is derived the following

(11.) GENERAL RULE. Arrange the coefficients of the given equation in a row, commencing with that of the first term.

Add the product of the first root figure, found by trial, and the first coefficient to the second coefficient; the product of the sum and same figure to the third coefficient, and so on to the last coefficient; and the last sum will be the divisor.

Repeat this process with the first coefficient and these sums, and the number under the last sum will be the trial divisor for the next figure.

Perform a similar process with the first coefficient and these second sums, stopping under the $n-1$th coefficient.

Perform again a similar process with the same first coefficient and these last sums, stopping here under the preceding, or $n-2$th coefficient, and so on till the last sum falls under the second coefficient.

Find now from the trial divisor the next figure of the root, and proceed with the last set of sums and this new figure exactly the same as with the original coefficients and the first figure in finding

the preceding divisor, and the next divisor will be obtained; and in a similar manner are the other divisors to be determined.

Note. After the first or second decimal of the root is obtained, the work of each column may be contracted, as in cubic equations.*

EXAMPLES.

1. Extract the root of the biquadratic equation $x^4 - 3x^2 + 75x = 10000$.

Here the first figure of the root is 9.

1	0	−3	75	10000 (9·886 root very near.
	9	81	702	6993
	9	78	777	3007
	9	162	2160	2678
	18	240	2937	329
	9	243	410	306
	27	483	334|7	23
	9	29	43	23
	3|6	51|2	378	
		3	4	
		5|4	3|8|2.	

The same equation solved by bringing down one period of decimals.

1	0	−3	75	10000 (9·8860027.
	9	81	702	6993
	9	78	777	3007
	9	162	2160	2677·5616
	18	240	2937	329·4384
	9	243	409·952	306·1662
	27	483	3346·952	23·2722
	9	29·44	434·016	23·2616
	36·8	512·44	3780·968	106
	·8	30·08	46·110	78
	37·6	542·52	3827·07|8	28
	·8	30·72	46·36	27
	38·4	573·24	3873·44	
	·8	3·14	3·50	
	3|9·|2	576·3|8	3876·9|4	
		3·1	3·5	
		579·|5	3|8|80·4	
		3		
		5|8|3.		

* In biquadratic, and higher equations, these contractions may always be made after the first decimal in the root is found.

Thus, we have found the root to eight places of figures, by bringing down only one period of decimals; if another period had been brought down, we should have found the root to be 9·88600270094, true to twelve places of figures.

2. Extract the root of the equation $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x = 110$.*

Here the first figure of the root is 4.

<table>
<tr><td>6</td><td>—10</td><td>—112</td><td>—207</td><td>110 (4·46410161</td></tr>
<tr><td>4</td><td>40</td><td>120</td><td>32</td><td>—700</td></tr>
<tr><td>10</td><td>30</td><td>8</td><td>—175</td><td>810</td></tr>
<tr><td>4</td><td>56</td><td>344</td><td>1408</td><td>667·05984</td></tr>
<tr><td>14</td><td>86</td><td>352</td><td>1233</td><td>142·94016</td></tr>
<tr><td>4</td><td>72</td><td>632</td><td>434·6496</td><td>133·46395</td></tr>
<tr><td>18</td><td>158</td><td>984</td><td>1667·6496</td><td>9·47621</td></tr>
<tr><td>4</td><td>88</td><td>102·624</td><td>477·4144</td><td>9·24089</td></tr>
<tr><td>22</td><td>246</td><td>1086·624</td><td>2145·0640</td><td>23532</td></tr>
<tr><td>4</td><td>10·56</td><td>106·912</td><td>79·3352</td><td>23158</td></tr>
<tr><td>26·4</td><td>256·56</td><td>1193·536</td><td>2224·399|2</td><td>374</td></tr>
<tr><td>·4</td><td>10·72</td><td>111·264</td><td>80·389</td><td>232</td></tr>
<tr><td>26·8</td><td>267·28</td><td>1304·800</td><td>2304·788</td><td>142</td></tr>
<tr><td>·4</td><td>10·88</td><td>17·453</td><td>5·434</td><td>139</td></tr>
<tr><td>27·2</td><td>278·16</td><td>1322·25|3</td><td>2310·22|2</td><td>3</td></tr>
<tr><td>·4</td><td>11·04</td><td>17.56</td><td>5·44</td><td>2</td></tr>
<tr><td>27·6</td><td>289·20</td><td>1339·8|1</td><td>2315·66</td><td></td></tr>
<tr><td>·4</td><td>1·68</td><td>17·6</td><td>14</td><td></td></tr>
<tr><td>2|8·|0</td><td>290·8|8</td><td>1357·4</td><td>2315·8|0</td><td></td></tr>
<tr><td></td><td>1·7</td><td>1·2</td><td>1</td><td></td></tr>
<tr><td></td><td>292·|6</td><td>1358·|6</td><td>2|3|1|5·9</td><td></td></tr>
<tr><td></td><td>1</td><td>1</td><td></td><td></td></tr>
<tr><td></td><td>|2|9|4</td><td>1|3|6|0.</td><td></td><td></td></tr>
</table>

SCHOLIUM.

Although only one root of the equation has been obtained in each of the above examples, yet either of the other possible roots may be had by a similar process; or the equation may be depressed a de-

* The *positive* root is here meant, as there are three negative roots to this equation, which are whole numbers, viz. —5, —3, and —1.

In the tract on the solution of equations by Mr. Holdred the discoverer of the above method, may be seen the solution to an equation as high as the fifteenth degree, to which the reader is referred, as the breadth of the page here will not admit of the insertion of the work requisite for an equation above the fifth degree.

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By the method here pointed out, all the operations are the following:

$-11-(-1-3)=-7=a:\dfrac{30}{3\times 1}=10=b$, $\therefore$ the quadratic is $x^2-7x+10=0$, which might readily have been ascertained mentally.

If only one root of a biquadratic equation be given, then the equation must be depressed to a cubic by performing the actual division; but this cubic may be depressed by the shorter method in Art. (6.)

THE END.